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Corporate Office:
Plot 99, Sector 44 Institutional Area,
Gurgaon -122 003 (HR), Tel : 0124-4951200
e-mail : info@mtg.in website : www.mtg.in

Regd. Office:
406, Taj Apartment, Near Safdarjung Hospital,
Ring Road, New Delhi - 110029.
Managing Editor : Mahabir Singh
Editor : Anil Ahlawat

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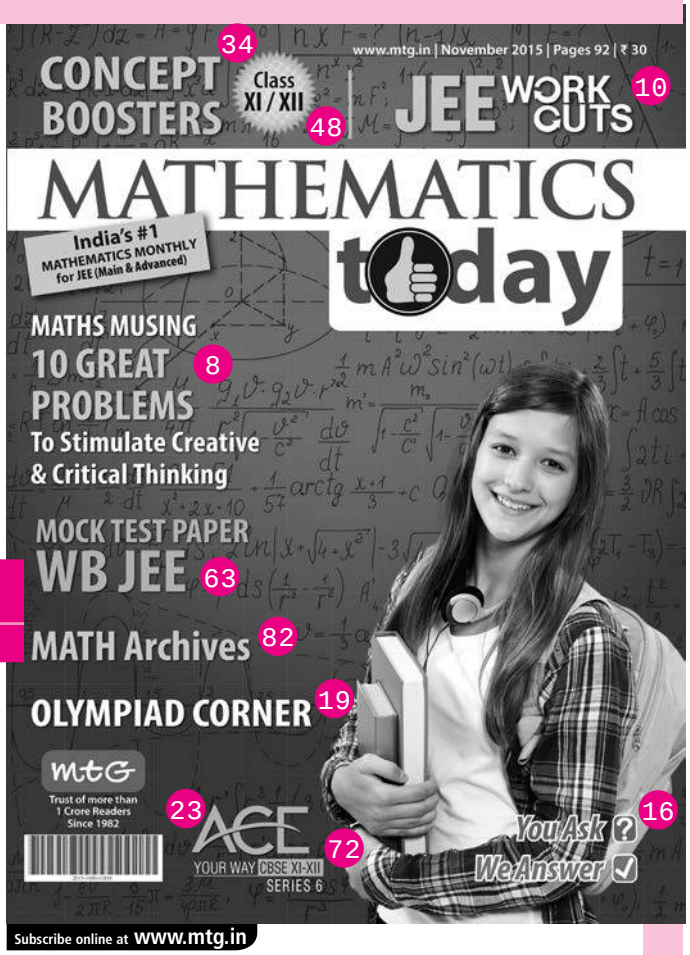

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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.
During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.
Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

Prof. Dr. Ramanaiah Gundala, Former Dean of Science and Humanities, Anna University, Chennai

PROBLEM Set 155

JEE MAIN

1. Let $0 < a < b < 125$. The number of pairs of integers (a, b), such that the A.M. of a and b exceeds their G.M. by 2, is
(a) 8 (b) 9 (c) 10 (d) 7

2. The area bounded by the curves $y = |1 - x^2|$ and $y = 5 - |x|$ is
(a) 18 (b) 19 (c) 20 (d) 21

3. Let $\{a, b\}$ be a subset of $\{1,2,3,\ldots,100\}$. The probability of $a^2 - b^2$ is divisible by 5 is
(a) $\frac{31}{99}$ (b) $\frac{32}{99}$ (c) $\frac{34}{99}$ (d) $\frac{35}{99}$

4. In a triangle ABC, if $A : B : C = 1 : 2 : 4$, then $\tan B \tan C + \tan C \tan A + \tan A \tan B =$
(a) -4 (b) -5 (c) -6 (d) -7

5. P is a point on the line $\vec{r} = 5\hat{i} + 7\hat{j} - 2\hat{k} + s(3\hat{i} - \hat{j} + \hat{k})$ and Q is a point on the line $\vec{r} = -3\hat{i} + 3\hat{j} + 6\hat{k} + t(-3\hat{i} + 2\hat{j} + 4\hat{k})$. If $\overrightarrow{PQ}$ is parallel to the vector $2\hat{i} + 7\hat{j} - 5\hat{k}$, then $|\overrightarrow{PQ}| =$
(a) $\sqrt{78}$ (b) $2\sqrt{78}$ (c) $4\sqrt{78}$ (d) $3\sqrt{78}$

JEE ADVANCED

6. The number of non-decreasing onto functions for $\{1, 2, 3, 4, 5, 6, 7, 8\}$ to $\{1, 3, 5, 7, 9\}$ is divisible by
(a) 2 (b) 3 (c) 5 (d) 7

COMPREHENSION

Let Z be a complex number such that $Z = re^{i\theta}$ and $\left|Z + \frac{1}{Z}\right| = 2$

7. If $\theta = \frac{\pi}{6}$, then $r =$
(a) $\frac{\sqrt{7}+\sqrt{3}}{2}$ (b) $\frac{\sqrt{3}+1}{\sqrt{2}}$
(c) $\frac{\sqrt{5}+1}{2}$ (d) $\frac{\sqrt{5}-1}{2}$

8. If $\theta = \frac{\pi}{3}$, then $r =$
(a) $\frac{\sqrt{7}+\sqrt{3}}{2}$ (b) $\frac{\sqrt{5}-1}{2}$
(c) $\frac{\sqrt{3}-1}{\sqrt{2}}$ (d) $\frac{\sqrt{7}-\sqrt{3}}{2}$

INTEGER MATCH

9. In a quadrilateral $ABCD$, $BC = 8$, $CD = 12$, $AD = 10$, $\angle A = \angle B = \frac{\pi}{3}$. If $AB = m + \sqrt{n}$, then the sum of the digits of $(m + n)$ is

MATCHING LIST

10.

	Column-I		Column-II
P.	If $\lim_{x\to 0}\left(1 + x + \frac{f(x)}{x}\right)^{1/x} = e^3$, then $\lim_{x\to 0}\frac{f(x)}{x^2} =$	1.	0
Q.	The number of roots of the equation $\tan x + \sec x = 2\cos x$ in $[0, 2\pi]$ is	2.	2
R.	Let $x^2 + y^2 - 4x - 2y - 11 = 0$ be a circle. The area of the quadrilateral formed by the tangents from the point (4, 5) with a pair of radii is	3.	4
S.	Let S and S_1 be the foci of the ellipse $9x^2 + 5y^2 = 30y$ and a point P be (3, 3). The area of triangle PSS_1 is	4.	6
		5.	8

	P	Q	R	S
(a)	2	2	5	4
(b)	1	1	3	4
(c)	5	4	3	2
(d)	4	2	3	5

See Solution set of Maths Musing 154 on page no. 18

Prof. Ramanaiah is the author of MTG JEE(Main & Advanced) Mathematics series

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JEE WORK OUTS

PAPER-1

SECTION-1

This section contains 8 questions. The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. Marking scheme : +4 If the bubble corresponding to the answer is darkened, 0 In all other cases.

1. The area bounded by $y = 2 - |2 - x|$ and $y = \frac{3}{|x|}$ is $\frac{k - 3\log_e(3)}{2}$, where k is

2. Let $f(x) = [x] + \{x\}^2$, $f: R \to R$, then area of figure bounded by $y = f^{-1}(x)$, $y = 0$ between the ordinates $x = 1/2$ and $x = 5$ is ($[\cdot]$ = G.I.F) and $\{\}$ = fractional function) $\frac{40\sqrt{2} - 1}{k\sqrt{2}}$, where k is

3. Area bounded by the curve $y = g(f(x))$, x-axis, $x = -3$ and $x = 4$, if $g(x) = e^x$, $f(x) = \{x\}$, is $k(e - 1)$, where k is

4. If $y = 2\sin(x) + \sin(2x)$, $0 \le x \le \pi$, then the area enclosed by the curve and the x-axis is

5. The value of $\int_1^{e^{37}} \frac{\pi \sin(\pi \ln x)}{x} dx$ is

6. If $A = \int_0^{3\pi/2} \frac{\cos x}{\cos x - \sin x} dx$ and $B = \int_0^{3\pi/2} \frac{\sin x}{\cos x - \sin x} dx$ then the value of $A + B$ is

7. The area bounded by $y = xe^{|x|}$ and lines $|x| = 1$, $y = 0$ is

8. If $\int_0^1 \frac{(1 - x^2)dx}{1 + x^2 + x^4} = \frac{1}{2}\ln(b)$, then find the value of b.

SECTION-2

This section contains TEN questions. Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option (s) is (are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is (are) darkened, 0 If none of the bubbles is darkened, 2 In all other cases.

9. If $\int \frac{xe^x}{\sqrt{(1 + e^x)}} dx = f(x)\sqrt{(1 + e^x)} - 2\ln g(x) + C$, then

(a) $f(x) = x - 1$ (b) $g(x) = \frac{\sqrt{1 + e^x} - 1}{\sqrt{1 + e^x} + 1}$

(c) $g(x) = \frac{\sqrt{1 + e^x} + 1}{\sqrt{1 + e^x} - 1}$ (d) $f(x) = 2(x - 2)$

10. If $I = \int \sec^2 x \operatorname{cosec}^4 x dx$
$= A\cot^3 x + B\tan x + C\cot x + D$, then

(a) $A = -1/3$ (b) $B = 2$

(c) $C = -2$ (d) none of these

11. A curve $g(x) = \int x^{27}(1 + x + x^2)^6(6x^2 + 5x + 4)dx$ is passing through origin, then

(a) $g(1) = \frac{3^7}{7}$ (b) $g(1) = \frac{2^7}{7}$

(c) $g(-1) = \frac{1}{7}$ (d) $g(-1) = \frac{3^7}{14}$

12. If $f(x) = \int \frac{x^8 + 4}{x^4 - 2x^2 + 2} dx$ and $f(0) = 0$, then

(a) $f(x)$ is an odd function

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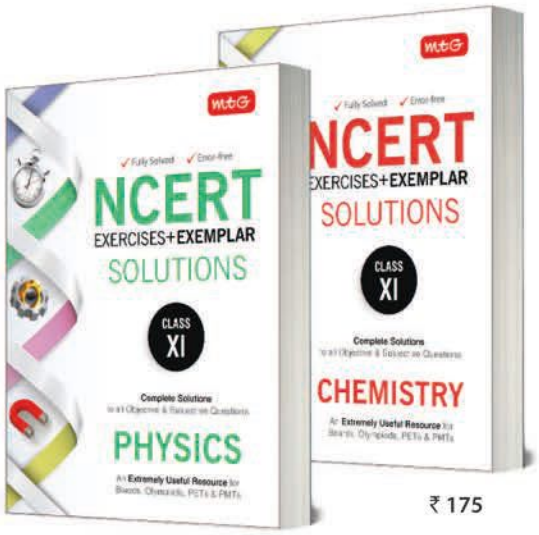

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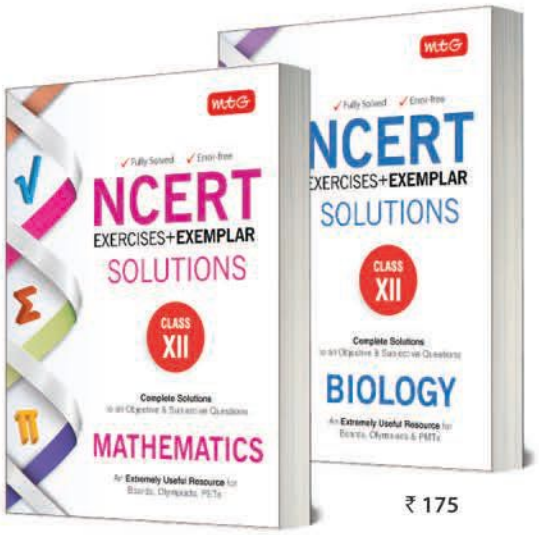

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(b) $f(x)$ has range R
(c) $f(x)$ has atleast one real root
(d) $f(x)$ is a monotonic function

13. Let $f'(x)=\frac{e^{\sin x}}{x}, x>0.$

If $\int_1^4 \frac{3e^{\sin(x^3)}}{x}dx = f(k)-f(l)$, then

(a) $k=64$ (b) $k=32$
(c) $l=2$ (d) $l=1$

14. If $f(x) = ae^{2x} + be^x + cx$ satisfies the conditions $f(0)=-1, f'(\log_e 2)=31, \int_0^{\log_e(4)} (f(x)-cx)dx=\frac{39}{2}$, then

(a) $a=5$ (b) $b=-6$
(c) $c=2$ (d) $a=3$

15. A function $y = f(x)$ satisfying the differential equation $\frac{dy}{dx}\cdot\sin x - y\cos x + \frac{\sin^2(x)}{x^2}=0$ is such that $y\to 0$ as $x\to\infty$ then

(a) $\lim_{x\to 0} f(x)=1$ (b) $\int_0^{\pi/2} f(x)dx<\frac{\pi}{2}$
(c) $\int_0^{\pi/2} f(x)dx>1$ (d) $f(x)$ is even

16. The function $f(x)=\int_0^x \sqrt{1-t^4}\,dt$ is such that

(a) It is defined on the interval [–1, 1]
(b) It is an increasing function
(c) It is an odd function
(d) The point (0, 0) is the point of inflection

17. If $I_1=\int_x^1 \frac{dt}{1+t^2}$ and $I_2=\int_1^{1/x}\frac{dt}{1+t^2}$ for $x>0$, then

(a) $I_1=I_2$ (b) $I_1>I_2$
(c) $I_2>I_1$ (d) $I_2=\cot^{-1}x-\frac{\pi}{4}$

18. Which of the following definite integrals when simplified reduces to zero?

(a) $\int_{\pi/5}^{3\pi/10} \ln\frac{4+3\sin x}{4+3\cos x}dx$ (b) $\int_0^{2\pi}\sin^4\frac{x}{2}\cos^5\frac{x}{2}dx$

(c) $\int_0^{\pi}\frac{\sin 4x}{\sin x}dx$ (d) $\int_0^{\infty} e^{-x}(\cos x-\sin x)dx$

SECTION-3

This section contains TWO questions. Each question contains two columns, Column I and Column II. Column I has four entries (A), (B), (C) and (D), Column II has five entries (P), (Q), (R), (S) and (T). Match the entries in Column I with the entries in Column II. One or more entries in Column I may match with one or more entries in Column II. The ORS contains a 4 × 5 matrix whose layout will be similar to the one shown below.

(A) P Q R S T
(B) P Q R S T
(C) P Q R S T
(D) P Q R S T

For each entry in Column I, darken the bubbles of all the matching entries. For example, if entry (A) in Column I matches with entries (Q), (R) and (T), then darken these three bubbles in the ORS. Similarly, for entries (B), (C) and (D). Marking scheme : For each entry in Column I, +2 If only the bubble(s) corresponding to all the correct match(es) is (are) darkened, 0 If none of the bubbles is darkened, –1 In all other cases.

19. Match the following:

Column I		Column II	
(A)	If $\int\frac{2^x}{\sqrt{1-4^x}}dx$ $=k\sin^{-1}(f(x))+C$, then k is greater than	p.	0
(B)	If $\int\frac{(\sqrt{x})^5}{(\sqrt{x})^7+x^6}dx$ $=a\ln\frac{x^k}{x^k+1}+c$, then ak is less than	q.	1
(C)	If $\int\frac{x^4+1}{x(x^2+1)^2}dx$ $=k\ln\lvert x\rvert+\frac{m}{1+x^2}+n$, where n is the constant of integration, then mk is greater than	r.	3
(D)	If $\int\frac{dx}{5+4\cos x}$ $=k\tan^{-1}\left(m\tan\frac{x}{2}\right)+C$ then k/m is greater than	s.	4

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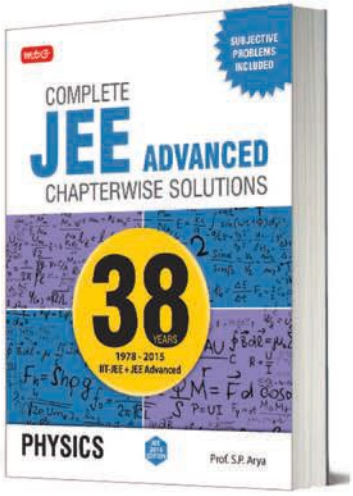

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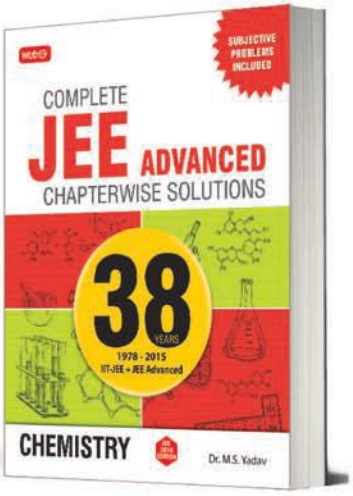

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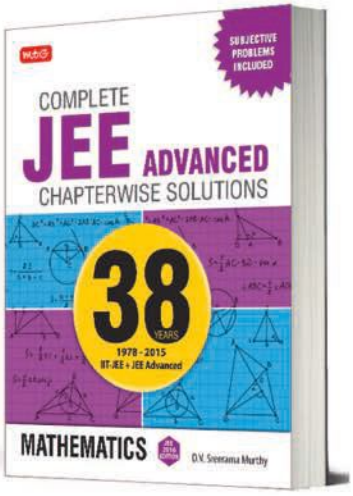

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20. Match the following:

Column I		Column II	
(A)	$\int_0^{\pi} x\ln(\sin x)dx =$	p.	$-\frac{\pi^2}{2}\ln(2)$
(B)	$\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)d\theta =$	q.	$\frac{\pi}{4}+\frac{1}{2}\ln 2$
(C)	$\int_0^{\pi/2} \frac{\cos x\,dx}{1+\cos x+\sin x} =$	r.	$\frac{\pi}{8}\ln 2$
(D)	$\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x\,dx}{(1-x^2)^{3/2}} =$	s.	$\frac{\pi}{4}-\frac{1}{2}\ln 2$

PAPER-2

SECTION-1

This section contains EIGHT questions. The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. Marking scheme : +4 If the bubble corresponding to the answer is darkened, 0 In all other cases.

1. The order of the differential equation of all circles, having centre on y-axis and passing through the origin is

2. Area bounded by $y = x\sin x$ and x-axis between $x = 0$ and $x = 2\pi$ is $k\pi$, where k is

3. The area enclosed between the curve $y = \log_e(x + e)$ and the coordinate axes is

4. Evaluate $\int_{1/2}^{2} \frac{1}{x}\text{cosec}^{101}\left(x-\frac{1}{x}\right)dx$

5. If $\int_0^{\pi} \frac{x^2\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx = \frac{k}{\pi}$, then k is

6. Sum of the two integral $\int_{-4}^{-5} e^{(x+5)^2}dx + 3\int_{1/3}^{2/3} e^{9(x-2/3)^2}dx$ is

7. If $\int_0^x f(t)dt = x + \int_x^1 tf(t)dt$, then find the value of $2f(1)$.

8. If $\lim_{x\to\infty} \frac{\int_0^x (\tan^{-1}t)^2dt}{\sqrt{1+x^2}} = \frac{\pi^2}{k}$, then k is

SECTION-2

This section contains EIGHT questions. Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option (s) is (are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is (are) darkened, 0 If none of the bubbles is darkened, –2 In all other cases.

9. If $\int \sin x\,d(\sec x) = f(x) - g(x) + c$, then

(a) $f(x) = \sec x$ (b) $f(x) = \tan x$

(c) $g(x) = 2x$ (d) $g(x) = x$

10. If $\int \frac{x^4+1}{x^6+1}dx = \tan^{-1}(f(x)) - \frac{2}{3}\tan^{-1}(g(x)) + C$, then

(a) both $f(x)$ and $g(x)$ are odd functions

(b) $f(x)$ is monotonic function

(c) $f(x) = g(x)$ has no real roots

(d) $\int \frac{f(x)}{g(x)}dx = -\frac{1}{x} + \frac{3}{x^3} + c$

11. If $\int \frac{x^2-x+1}{(x^2+1)^{\frac{3}{2}}}e^x dx = e^x f(x) + c$, then

(a) $f(x)$ is an even function

(b) $f(x)$ is a bounded function

(c) The range of $f(x)$ is $(0, 1]$

(d) $f(x)$ has two points of extrema

12. The solution of differential equation $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx} = 0$ is

(a) $2xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + k$

(b) $xe^{2\tan^{-1}y} = e^{2\tan^{-1}y} + k$

(c) $xe^{\tan^{-1}y} = \tan^{-1}y + k$

(d) $(x-2) = ke^{\tan^{-1}y}$

13. Which of the following is the integrating factor of $x\log x\frac{dy}{dx} + y = 2\log x$?

(a) x (b) e^x

(c) $\log_e(x)$ (d) $\log_e(\log_e x)$

14. The value of $\int_{-2}^{2} \min(x-[x], -x-[-x])dx$ is ([·] denotes the greatest integer function)
(a) 0 (b) 1
(c) 2 (d) none of these

15. Let $f(a) > 0$, and let $f(x)$ be a non decreasing continuous function in $[a, b]$. Then $\frac{1}{b-a}\int_a^b f(x)dx$ is
(a) less than equal to $f(b)$
(b) greater than equal to $f(a)$
(c) maximum value $bf(b)$
(d) minimum value $\frac{f(a)}{b-a}$

16. The values of α which satisfy $\int_{\pi/2}^{\alpha} \sin x dx = \sin 2\alpha$ $(\alpha \in [0, 2\pi])$ are equal to
(a) $\pi/2$ (b) $3\pi/2$ (c) $7\pi/6$ (d) $11\pi/6$

SECTION-3

This section contains TWO paragraphs. Based on each paragraph, there will be TWO questions. Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option(s) is(are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened, 0 If none of the bubbles is darkened, -2 In all other cases

Paragraph for Q. No. 17 & 18

An even function f is defined and integrable everywhere and is periodic with period 2.

Also function $g(x) = \int_0^x f(t)dt$ and $g(1) = A$.

17. Function $g(x)$ is
(a) odd (b) even
(c) neither even nor odd
(d) can't be determined

18. Value of $g(2)$ in terms of A is
(a) $2A$ (b) $A/2$ (c) $4A$ (d) $A/4$

Paragraph for Q. No. 19 & 20

Sometimes equations which are not linear can be reduced to the linear form by suitable transformations.

For example an equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x only or constants and n ($\neq 0$ and 1) is a constant.

$\because \frac{dy}{dx} + Py = Qy^n \Rightarrow y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$ ... (i)

[Dividing by y^n on both sides]

Let $y^{1-n} = z \therefore (1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

Put the value of $y^{-n}\frac{dy}{dx}$ in (i), we get

$\frac{dz}{dx} + (1-n)Pz = (1-n)Q$ which is linear in z.

So, solution is

$z \cdot e^{\int (1-n)Pdx} = C + \int (1-n)Q \cdot e^{\int (1-n)Pdx} dx$

i.e. $y^{1-n} e^{\int (1-n)Pdx} = C + \int (1-n)Q \cdot e^{\int (1-n)Pdx} dx$

This form is called Bernoulli's equation.

19. Solution of $\frac{dy}{dx} = \frac{y\psi'(x) - y^2}{\psi(x)}$, where $\psi(x)$ is a function of x is
(a) $\psi(x) = x + C$ (arbitrary constant)
(b) $\frac{\psi(x)}{y} = x + C$ (arbitrary constant)
(c) $y\psi(x) = -x + C$ (arbitrary constant)
(d) $y\psi(x) = x + C$ (arbitrary constant)

20. Solution of the differential equation

$\frac{dy}{dx} = \frac{1}{xy[x^2 \sin y^2 + 1]}$ is

(a) $e^{y^2}\left[\frac{1}{2}(\cos y^2 - \sin y^2) - \frac{1}{x^2}\right] = C$

(b) $e^{y^2}\left[\frac{1}{2}(\cos y^2 + \sin y^2) - \frac{1}{x^2}\right] = C$

(c) $e^{y^2}\left[\frac{1}{2}(\cos y^2 - \sin y^2) + \frac{1}{x^2}\right] = C$

(d) $e^{y^2}\left[\frac{1}{2}(\cos y^2 + \sin y^2) + \frac{1}{x^2}\right] = C$

ANSWERS KEY

PAPER-1

1.	(4)	2.	(3)	3.	(7)	4.	(4)	5.	(2)
6.	(0)	7.	(2)	8.	(3)	9.	(b, d)	10.	(a, c)
11.	(a, c)	12.	(a, b, c, d)			13.	(a, d)	14.	(a, b)
15.	(a, b, c, d)			16.	(a, b, c, d)			17.	(a, d)

18. (a, b, c, d)
19. (A) → (p, q); (B) → (r, s); (C) → (p); (D) → (p, q)
20. (A) → (p), (B) → (r), (C) → (s), (D) → (s)

PAPER-2

1.	(1)	2.	(4)	3.	(1)	4.	(0)	5.	(8)
6.	(0)	7.	(1)	8.	(4)	9.	(b, d)	10.	(a, c, d)
11.	(a, b, c)			12.	(a)	13.	(c)	14.	(b)
15.	(a, b)	16.	(a, b , c, d)			17.	(a)	18.	(a)

19. (b) 20. (a)

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1. Integrate $\int \tan(x-\alpha)\cdot\tan(x+\alpha)\cdot\tan 2x\,dx$

– Rakesh Avasthi, Patna

Ans. $\tan 2x = \tan(\overline{x+\alpha}+\overline{x-\alpha})$

$$= \frac{\tan(x+\alpha)+\tan(x-\alpha)}{1-\tan(x+\alpha)\tan(x-\alpha)}$$

or $\tan 2x - \tan 2x\cdot\tan(x+\alpha)\tan(x-\alpha)$

$$= \tan(x+\alpha)+\tan(x-\alpha)$$

$\therefore$ $\tan(x-\alpha)\tan(x+\alpha)\tan 2x$

$$= \tan 2x - \tan(x+\alpha) - \tan(x-\alpha)$$

Let $I = \int \tan(x-\alpha)\tan(x+\alpha)\tan 2x\,dx$

$\therefore$ $I = \int \{\tan 2x - \tan(x+\alpha) - \tan(x-\alpha)\}\,dx$

$$= \frac{1}{2}\log\sec 2x + \log\cos(x+\alpha) + \log\cos(x-\alpha) + c$$

$$= \log\left\{\sqrt{\sec 2x}\cdot\cos(x+\alpha)\cdot\cos(x-\alpha)\right\} + c.$$

2. If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$ (where $pq \neq 0$) are bisected by the x-axis, then prove that $p^2 > 8q^2$

– Nalini Sharma, Bangalore

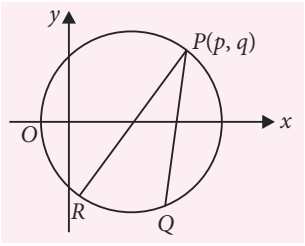
Ans. Let PQ be a chord of the given circle passing through $P(p, q)$ and the coordinates of Q be (x, y). Since PQ is bisected by the x-axis, the mid-point of PQ lies on the x-axis which gives $y = -q$.

Now Q lies on the circle $x^2 + y^2 - px - qy = 0$

so $x^2 + q^2 - px + q^2 = 0$

$\Rightarrow$ $x^2 - px + 2q^2 = 0$ …(i)

which gives two values of x and hence the coordinates of two points Q and R (say), so that the chords PQ and PR are bisected by x-axis. If the chords PQ and PR are distinct, the roots of (i) are real and distinct.



$\Rightarrow$ the discriminant $p^2 - 8q^2 > 0$

$\Rightarrow$ $p^2 > 8q^2$.

3. If $S_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$ and $t_n = \sum_{r=0}^{n} \frac{r}{{}^nC_r}$, then prove that $\frac{t_n}{S_n} = \frac{n}{2}$

– Ramesh Ahuja, Ludhiana

Ans. Given, $S_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$ and $t_n = \sum_{r=0}^{n} \frac{r}{{}^nC_r}$

$\Rightarrow$ $S_n = \sum_{r=0}^{n} \frac{1}{{}^nC_{n-r}}$

$\Rightarrow$ $nS_n = \sum_{r=0}^{n} \frac{n}{{}^nC_{n-r}}$

$\Rightarrow$ $nS_n = \sum_{r=0}^{n} \left[\frac{n-r}{{}^nC_{n-r}} + \frac{r}{{}^nC_{n-r}}\right]$

$\Rightarrow$ $nS_n = \sum_{r=0}^{n} \left[\frac{n-r}{{}^nC_{n-r}}\right] + \sum_{r=0}^{n} \left[\frac{r}{{}^nC_r}\right]$

$\Rightarrow$ $nS_n = \left[\frac{n}{{}^nC_n} + \frac{n-1}{{}^nC_{n-1}} + \ldots + \frac{1}{{}^nC_1}\right] + \sum_{r=0}^{n} \frac{r}{{}^nC_r}$

$\Rightarrow$ $nS_n = t_n + t_n$

$\Rightarrow$ $\frac{t_n}{S_n} = \frac{n}{2}$

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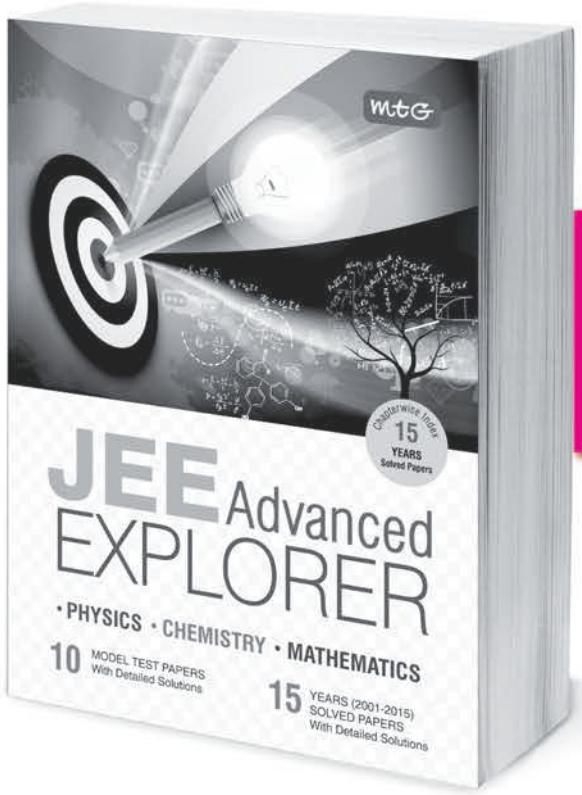

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MATHS MUSING

SOLUTION SET 154

1. **(c) :** Let . stand for H or T. The 4 consecutive heads start with 1^{st}, 2^{nd}, 3^{rd}, ... 7^{th} toss

HHHH....., THHHH .....

.THHHH ....., ..THHHH.....

....THHHH

$$\text{Prob.} = \frac{1}{2^4} + \frac{6}{2^5} = \frac{1}{4}$$

2. **(d) :** $f(-\theta) = f(\theta),\ f(\theta + \pi) = f(\theta)$

$\therefore f(\theta)$ is an even function periodic with period π.

Maximum value $= f(0) = 1 + \sin^2 1$

$$\text{Minimum value} = f\left(\frac{\pi}{2}\right) = \cos^2 1$$

The difference $= 1 + \sin^2 1 - \cos^2 1 = 1 - \cos 2$

3. **(b) :** Additional digits 0, 4 or 1, 3 or 2, 2

$$00124 \rightarrow 3 \times \frac{4!}{2!} = 36 \text{ numbers}$$

$$001123 \rightarrow 4! + 2 \times \frac{4!}{2!} = 48 \text{ numbers}$$

$$01222 \rightarrow \frac{4!}{3!} + \frac{4!}{2!} = 16 \text{ numbers}$$

Total numbers $= 36 + 48 + 16 = 100$

4. **(c) :** Let $c = kb$, 20, b, kl are in H.P.

$$\Rightarrow \quad b = 40 - \frac{20}{k},\ k = 2, 4, 5, 10, 20 \rightarrow$$

$b = 30, 35, 36, 38, 39$

$\therefore$ Number of value of $b = 5$

5. **(b) :** Let $x + y = z \rightarrow dx = \left(1 - \frac{2}{z+2}\right)dz$

$x + c_1 = z - 2\ln(z + 2)$

$\Rightarrow x + y + z = ce^{y/2}$

6. **(a,b) :** $p = (x_1, y_1) \rightarrow t_1^2 = x_1^2 + y_1^2 - a^2,\ t_2^2 = x_1^2 + y_1^2 - 2ax_1,\ t_3^2 = x_1^2 + y_1^2 - 2ay_1,\ t_1^4 = t_2^2 t_3^2 + a^4 \rightarrow$ locus of p is $(x + y)(x^2 + y^2 - ax - ay) = 0$, straight line and circle.

7. **(b) :** $f(x + y) = e^x f(y) + e^y f(x)$

$x = y = 0 \rightarrow f(0) = 0$

$$\frac{\partial}{\partial x} f(x+y) = \frac{\partial}{\partial y} f(x+y)$$

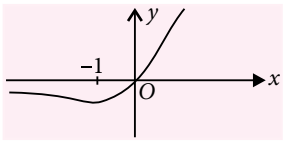
$$\Rightarrow \frac{f'(x) - f(x)}{e^x} = \frac{f'(y) - f(y)}{e^y} = k, \text{ constant}$$

$\therefore f'(x) - f(x) = ke^x \rightarrow f(x) = (kx + c)e^x$

$f(0) = 0 \rightarrow c = 0, f'(0) = 1 \rightarrow k = 1, f(x) = xe^x$

It has mininum at $x = -1$.

8. **(a) :** Area $= \int_{-\infty}^{0} -xe^x dx = 1$



9. **(9) :** $16\Delta^2 = (a + b + c)(b + c - a)(c + a - b)(a + b - c)$

$= [(b + c)^2 - a^2]\ [a^2 - (b - c)^2]$

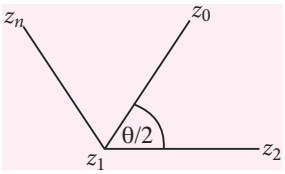
$= (9b^2 - 27)\ (27 - b^2)$

$= 9(b^2 - 3)\ (27 - b^2)$

$$\frac{16\Delta^2}{9} = -81 + 30b^2 - b^4 = 144 - (b^2 - 15)^2 \le 144$$

$\therefore \quad \Delta \le 9$, maximum area $= 9$

10. **(b) :** P $\rightarrow$ 4, Q $\rightarrow$ 5, R $\rightarrow$ 3, S $\rightarrow$ 2



$\angle z_n z_1 z_2 = \theta$

$$z_0 - z_1 = (z_2 - z_1)\left|\frac{z_0 - z_1}{z_2 - z_1}\right| \text{cis}\,\frac{\theta}{2}$$

$$= \frac{1}{2}\left(1 + i\tan\frac{\theta}{2}\right)(z_2 - z_1)$$

$$\therefore \quad z_0 = Az_1 + \bar{A}z_2,\ A = \frac{1}{2}\left(1 - i\tan\frac{\theta}{2}\right)$$

$$|A|^2 = \frac{1}{2(1 + \cos\theta)}$$

$$n = 6, |A|^2 = 1,\ n = 8, |A|^2 = 1 + \frac{1}{\sqrt{2}}$$

$$n = 10, |A|^2 = \frac{3 + \sqrt{5}}{2},\ n = 12, |A|^2 = 2 + \sqrt{3}$$

Solution Sender of Maths Musing

SET-153

1. Harpal Singh (Punjab)
2. Gouri Sankar Adhikari (W.B.)
3. Khokon Kumar Nandi (W.B.)
4. Gajula Ravinder (Karimnagar)
5. N. Jayanthi (Hyderabad)

SET-154

1. Jayanthi (Hyderabad)
2. Khokon Kumar Nandi (W.B.)
3. Gouri Sankar Adhikari (W.B.)

OLYMPIAD CORNER

1. Prove that

$$\frac{a+b+c}{3} \le \frac{1}{4}\sqrt[3]{\frac{(b+c)^2(c+a)^2(a+b)^2}{abc}},$$

where $a, b, c > 0$. Equality holds if $a = b = c$.

2. A hexagon is inscribed in a circle with radius r. Two of its sides have length 1, two have length 2 and the last two have length 3. Prove that r is a root of the equation $2r^3 - 7r - 3 = 0$.

3. Prove that
$\sin^2(\theta + \alpha) + \sin^2(\theta + \beta) - 2\cos(\alpha - \beta)\sin(\theta + \alpha)\sin(\theta + \beta) = \sin^2(\alpha - \beta)$. …(1)

4. $ABCD$ is a rhombus with $\angle A = 60°$. Suppose that E, F are points on the sides AB, AD, respectively and that CE, CF, meet BD at P, Q respectively. Suppose that $BE^2 + DF^2 = EF^2$.
Prove that $BP^2 + DQ^2 = PQ^2$

5. The numbers 2, 4, 8, 16, ..., 2^n are written on a chalkboard. A student selects any two numbers a and b, erases them and replaces them by their average, namely $(a + b)/2$. She performs this operation $(n - 1)$ times until only one number is left. Let S_n and T_n denote the maximum and minimum possible value of this final number, respectively. Determine a formula for S_n and T_n in terms of n.

SOLUTIONS

1. By the arithmetic-geometric mean inequality, we have

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 \ge 6\sqrt[6]{a^6b^6c^6} = 6abc,$$

which implies

$$9(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 2abc)$$
$$\ge 8(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 3abc),$$
$$\text{or } 9(a + b)(b + c)(c + a) \ge 8(a + b + c)(ab + bc + ca)$$
$$= 4(a + b + c)(a(b + c) + b(c + a) + c(a + b)).$$

Using the arithmetic-geometric mean inequality again, we then have

$$\frac{3}{4}(a+b)(b+c)(c+a)$$
$$\ge (a+b+c).\frac{a(b+c)+b(c+a)+c(a+b)}{3}$$
$$\ge (a+b+c)\sqrt{abc(a+b)(b+c)(c+a)} \quad \ldots(i)$$

From (i), it follows that

$$\frac{1}{4}\sqrt[3]{\frac{(a+b)^2(b+c)^2(c+a)^2}{abc}} \ge \frac{a+b+c}{3}.$$

2. Equal chords subtend equal angles at the centre of a circle; if each of the sides of length i subtends an angle α_i $(i = 1, 2, 3)$ at the centre of the given circle, then $2\alpha_1 + 2\alpha_2 + 2\alpha_3 = 360°$,

whence $\frac{\alpha_1}{2} + \frac{\alpha_2}{2} = 90° - \frac{\alpha_3}{2}$,

and $\cos\left(\frac{\alpha_1}{2} + \frac{\alpha_2}{2}\right) = \cos\left(90° - \frac{\alpha_3}{2}\right) = \sin\frac{\alpha_3}{2}$.

Next we apply the addition formula for the cosine:

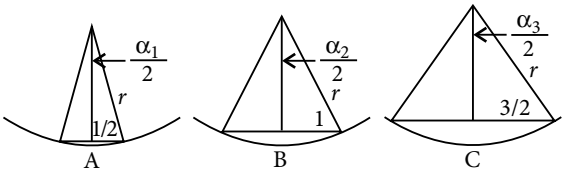
$$\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} - \sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2} = \sin\frac{\alpha_3}{2}, \quad \ldots(i)$$

where (see figures)

$$\sin\frac{\alpha_1}{2} = \frac{1/2}{r},\ \cos\frac{\alpha_1}{2} = \frac{\sqrt{4r^2-1}}{2r},$$

$$\sin\frac{\alpha_2}{2} = \frac{1}{r},\ \cos\frac{\alpha_2}{2} = \frac{\sqrt{r^2-1}}{r};$$

$$\sin\frac{\alpha_3}{2} = \frac{3/2}{r}$$



We substitute these expressions into (i) and obtain, after multiplying both sides by $2r^2$,

$$\sqrt{4r^2-1}\cdot\sqrt{r^2-1}-1=3r$$

Now write it in the form

$$\sqrt{(4r^2-1)(r^2-1)}=3r+1,$$

and square, obtaining

$$(4r^2-1)(r^2-1)=9r^2+6r+1,$$

which is equivalent to $r(2r^3-7r-3)=0$.

Since $r\neq 0$, we have $2r^3-7r-3=0$,

which was to be shown.

3. $2(\sin^2(\theta+\alpha)+\sin^2(\theta+\beta))$

$=2-(\cos(2\theta+2\alpha)+\cos(2\theta+2\beta))$

$=2-2\cos(2\theta+\alpha+\beta)\cos(\alpha-\beta)$,

$-4\cos(\alpha-\beta)\sin(\theta+\alpha)\sin(\theta+\beta)$

$=-2\cos(\alpha-\beta)(\cos(\alpha-\beta)-\cos(2\theta+\alpha+\beta))$,

so that, 2 × L.H.S. of (1)

$=2(1-\cos(2\theta+\alpha+\beta)\cos(\alpha-\beta)-\cos^2(\alpha-\beta)+$

$\cos(\alpha-\beta)\cos(2\theta+\alpha+\beta))$

$=2(1-\cos^2(\alpha-\beta))=2\sin^2(\alpha-\beta)$.

4. Let $AB=1$ and put $EB=x$ and $FD=y$

Then $DB=1$, $AE=1-x$ and $AF=1-y$

By applying the Cosine Law to ΔAEF ($\angle EAF=60°$) we transform the condition $x^2+y^2=EF^2$ to

$$x^2+y^2=(1-x)^2+(1-y)^2-2\cdot\frac{1}{2}\cdot(1-x)(1-y),$$

which simplifies to $1-x-y-xy=0$,

and so, $y=\dfrac{1-x}{1+x}$ …(i)

Furthermore, BP is the angle-bisector of $\angle EBC$, in ΔEBC so, by a well-known formula,

$$BP=\frac{2EB\cdot BC\cdot\cos 60°}{EB+BC}=\frac{x}{x+1}.$$

Similarly, $DQ=\dfrac{y}{y+1}$. Using (i), $DQ=\dfrac{1-x}{2}$.

Now, $PQ=DB-BP-DQ$

$$=1-\frac{x}{x+1}-\frac{1-x}{2}=\frac{1+x^2}{2(1+x)}.$$

Finally,

$$BP^2+DQ^2=\left(\frac{x}{x+1}\right)^2+\left(\frac{1-x}{2}\right)^2$$

$$=\frac{1+2x^2+x^4}{4(1+x)^2}=\frac{(1+x^2)^2}{4(1+x)^2}=PQ^2,$$

which completes the proof.

5. We shall prove by induction that if we perform the operation on the numbers $a_1, a_2, \ldots, a_{n-1}, a_n$, where $a_1<a_2<\ldots<a_n$, then

$$T_n=\frac{a_1}{2}+\frac{a_2}{4}+\frac{a_3}{8}+\ldots+\frac{a_{n-2}}{2^{n-2}}+\frac{a_{n-1}}{2^{n-1}}+\frac{a_n}{2^{n-1}}.$$

For $n=2$, we have two numbers, a_1 and a_2 and thus T_2 must be $(a_1+a_2)/2$. Thus the claim holds for $n=2$.

Assume that for $n=2, 3, \ldots, k-2, k-1$, the identity holds. Then for the case $n=k$, that is, when we start off with the numbers $a_1, a_2, a_3, \ldots, a_k$, we will perform the operation $k-2$ times and be left with two numbers, x and y. Say x comes from performing the operation on a set of p numbers from the set. They comes from performing the operation on the remaining set of $k-p$ numbers.

To illustrate this, say we start off with 2, 4, 8, 16, and 32. Then we can replace 2 and 8 by 5, 16 and 32 by 24, then 4 and 24 by 14. Then we are left with two numbers, 5 and 14. We got 5 from performing the operation on the numbers 2 and 8, and we got 14 from performing the operation on the other three numbers, namely 4, 16, and 32.

Let $b_1, b_2, b_3, \ldots, b_p$ be the set of numbers that we used to get the number x, where the b_i's are in increasing order and let $c_1, c_2, c_3, \ldots, c_{k-p}$ be the other numbers from the set that we used to get y, where the c_i's are in increasing order. Then, by our induction hypothesis, we have

$$x\geq\frac{b_1}{2}+\frac{b_2}{4}+\ldots.+\frac{b_{p-2}}{2^{p-2}}+\frac{b_{p-1}}{2^{p-1}}+\frac{b_p}{2^{p-1}},$$

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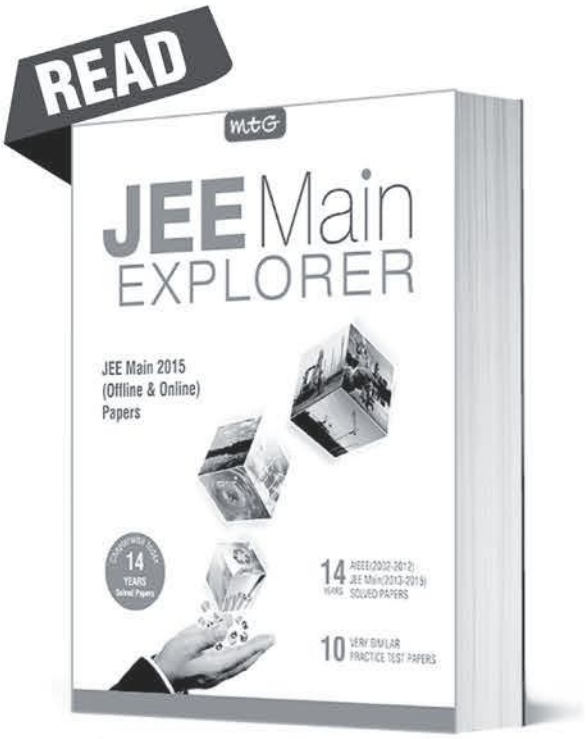

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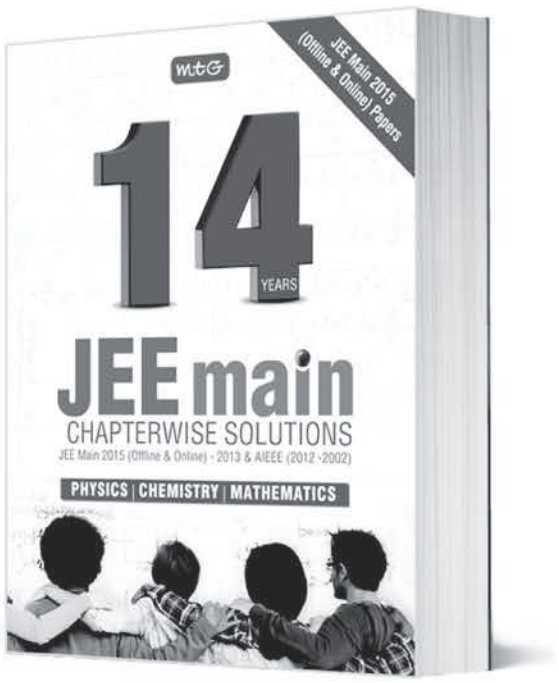

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and similarly,

$$y \geq \frac{c_1}{2} + \frac{c_2}{4} + ... + \frac{c_{k-p-2}}{2^{k-p-2}} + \frac{c_{k-p-1}}{2^{k-p-1}} + \frac{c_{k-p}}{2^{k-p-1}}.$$

Note that the b_i's and the c_i's are just some permutation of the set a_1, a_2, a_3, ..., a_k. Hence, to minimize the value of $(x + y)/2$, the average of the two numbers will be minimized when the denominators are as large as possible. Hence, we want either p or $k - p$ to be 1, since we will then have a denominator of 2^{k-1} in one of the terms.

Without loss of generality, assume that $k - p = 1$. Thus, we want to now show that

$$\frac{x+y}{2} \geq \frac{c_1}{2} + \frac{b_1}{4} + \frac{b_2}{8} + ... + \frac{b_{k-3}}{2^{k-2}} + \frac{b_{k-2}}{2^{k-1}} + \frac{b_{k-1}}{2^{k-1}}$$

$$\geq \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + ... + \frac{a_{k-2}}{2^{k-2}} + \frac{a_{k-1}}{2^{k-1}} + \frac{a_k}{2^{k-1}},$$

because this will prove the case for $n = k$.

Note that the b_i's are in increasing order, so if $c_1 = a_r$ for some r, then $b_1 = a_1$, $b_2 = a_2$, ..., $b_{r-1} = a_{r-1}$, $b_r = a_{r+1}$, $b_{r+1} = a_{r+2}$, . . ., $b_{k-1} = a_k$.

Noting that the a_i's are in increasing order, we have

$$\frac{a_r}{2} = \frac{a_r}{4} + \frac{a_r}{8} + \frac{a_r}{16} + ...$$

$$\geq \frac{a_r}{4} + \frac{a_r}{8} + \frac{a_r}{16} + ... + \frac{a_r}{2^r} \geq \frac{a_1}{4} + \frac{a_2}{8} + \frac{a_3}{16} + ... + \frac{a_{r-1}}{2^r}$$

Adding

$$\frac{a_1}{4} + \frac{a_2}{8} + \frac{a_3}{16} + ... + \frac{a_{r-1}}{2^r} + \frac{a_{r+1}}{2^{r+1}} + \frac{a_{r+2}}{2^{r+2}} + ... + \frac{a_{k-1}}{2^{k-1}} + \frac{a_k}{2^{k-1}}$$

to both sides, we get the desired inequality.

Thus, the induction is proved and hence, for our question, we take $a_i = 2^i$ for $i = 1, 2, ..., k$ and we have

$$T_n = \frac{2}{2} + \frac{4}{4} + \frac{8}{8} + ... + \frac{2^{n-2}}{2^{n-2}} + \frac{2^{n-1}}{2^{n-1}} + \frac{2^n}{2^{n-1}}$$

$$= (n - 1) \cdot 1 + 2 = n + 1$$

Thus, our formula for T_n is obtained by taking the a_i's in increasing order. In contrast, to get S_n, we want to take the a_i's in decreasing order, namely $a_1 = 2^n$, $a_2 = 2^{n-1}$, . . . , $a_{n-1} = 4$, $a_n = 2$.

This is so that we can get the maximum possible value at the end. Thus, we have

$$S_n = \frac{2^n}{2} + \frac{2^{n-1}}{4} + \frac{2^{n-2}}{8} + ... + \frac{2^3}{2^{n-2}} + \frac{2^2}{2^{n-1}} + \frac{2^1}{2^{n-1}}$$

$$= 2^{n-1} + 2^{n-3} + 2^{n-5} + ... + 2^{5-n} + 2^{3-n} + 2^{2-n}$$

$$= 2^{3-n} \cdot (1 + 2^2 + 2^4 + ... + 2^{2n-4}) + 2^{2-n}$$

$$= 2^{3-n} \cdot \frac{4^{n-1} - 1}{3} + 2^{2-n}$$

$$= \frac{8(4^{n-1} - 1)}{3 \cdot 2^n} + \frac{4}{2^n} = \frac{2^{2n+1} + 4}{3 \cdot 2^n}$$

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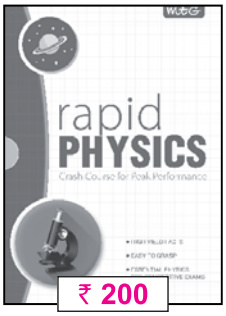

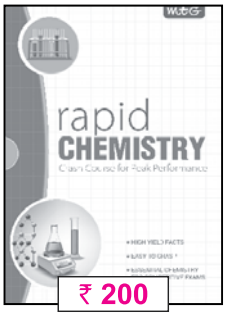

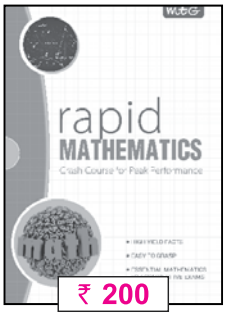

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Series 6

STRAIGHT LINES, CONIC SECTIONS AND INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

STRAIGHT LINES

DEFINITION

A path traced by a point in a constant direction and endlessly in its opposite direction is known as a straight line.

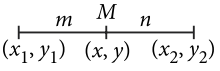
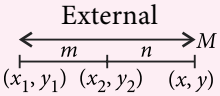
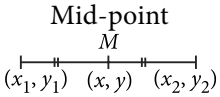
Distance Formula

Distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Section Formula

Let a point M divides a line segment joining the points (x_1, y_1) and (x_2, y_2) in the ratio $m : n$, then

Types of division	Coordinates of M
Internal 	$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$
External 	$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$
Mid-point 	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Area of Triangle

Area of ΔABC whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given by

$$\Delta = \frac{1}{2} \cdot |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| \text{ sq. unit.}$$

Note : The points A, B, C are collinear $\Leftrightarrow$ area of $\Delta ABC = 0$.

SLOPE OR GRADIENT OF A LINE

If θ is the inclination of a non-vertical line, then $m = \tan\theta$ is called the slope or gradient of the line.

θ is measured from x-axis and it is taken as positive or negative according as it is measured in anticlockwise or clockwise direction.

Note :

- The inclination of a line parallel to the x-axis or of a horizontal line is 0°, *i.e.* $\theta = 0°$. Hence, slope of a line parallel to x-axis is 0.
- The inclination of a line parallel to the y-axis or of a vertical line is 90°. *i.e.*, $\theta = 90°$. Hence, slope of a line perpendicular to y-axis is not defined.

Slope of a Line Passing through Two Given Points

Slope m of a non-vertical line passing through the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$.

Condition for Parallelism and Perpendicularity

Let L_1 and L_2 be two lines whose slopes are m_1 and m_2 respectively. Then,

(i) $L_1 \| L_2 \Leftrightarrow m_1 = m_2$

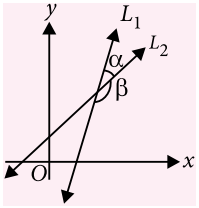
(ii) $L_1 \perp L_2 \Leftrightarrow m_1 m_2 = -1$

ANGLE BETWEEN TWO LINES

If α is the acute angle between two non-vertical lines L_1 and L_2 with slopes m_1 and m_2, then

$$\tan\alpha = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|,$$

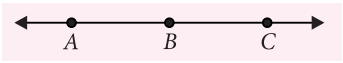
where $1 + m_1 m_2 \neq 0$.



Case I : If $\tan\alpha > 0$, then α is the acute angle between two lines and β is the obtuse angle.

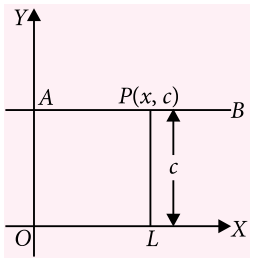
Case II : If $\tan\alpha < 0$, then α is the obtuse angle between two lines and β is the acute angle.

- **Collinearity of three points :** Three points A, B, C in a plane are said to be collinear, if and only if slope of AB = slope of BC.

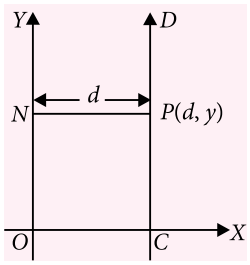


VARIOUS FORMS OF THE EQUATION OF A LINE

- **Equation of the x-axis :** Since ordinate of any point on the x-axis is zero. Hence, the equation of the x-axis is $y = 0$
- **Equation of the y-axis :** Since abscissa of any point on the y-axis is zero. Hence, the equation of the y-axis is $x = 0$.
- **Equation of a line parallel to the x-axis :** Let AB be any line parallel to the x-axis and at a distance c from it. The ordinate of any point P on the line AB is c. Hence, the equation of a line parallel to the x-axis is of the form $y = c$.



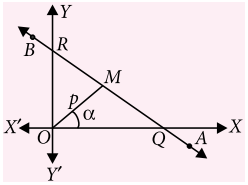
- **Equation of a line parallel to the y-axis :** Let CD be any line parallel to the y-axis and at a distance d from it. The abscissa of any point P on the line CD is d. Hence, the equation of a line parallel to the y-axis is of the form $x = d$.



- **Slope-intercept form :** The equation of the straight line whose slope is m and which cuts an intercept c on the y-axis *i.e.* which passes through the point $(0, c)$ is $y = mx + c$.

Note : If the line passes through the origin, then $c = 0$ and hence equation of the line will become $y = mx$.

- **Point slope form :** The equation of the straight line having slope m and passing through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.
- **Two-point form :** The equation of the straight line which passes through the point (x_1, y_1) and (x_2, y_2) is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.
- **Intercept form :** The equation of the straight line making intercepts a and b on x- axis and y-axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$
- **Normal form :** The equation of the straight line upon which the length of the perpendicular from the origin is p and this normal makes an angle α with the positive direction of x-axis is $x\cos\alpha + y\sin\alpha = p$



- **Distance form or parametric form or symmetric form :** The equation of the straight line passing through the point (x_1, y_1) and making an angle θ with the positive direction of x-axis is $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$

GENERAL EQUATION OF A LINE

Any equation of the form $Ax + By + C = 0$, where A, B and C are real constants such that A and B are not simultaneously zero is called general equation of the line.

Reduction of the general equation to the standard form:

(i) Reduction to slope-intercept form : If $B \neq 0$, then the equation $Ax + By + C = 0$ can be written as $y = -\frac{A}{B}x - \frac{C}{B}$, which is of the form $y = mx + c$ whose slope, $m = -\frac{A}{B}$ and y- intercept, $c = -\frac{C}{B}$

(ii) Reduction to intercept form : If $C \neq 0$, then the equation $Ax + By + C = 0$ can be written as

$\frac{x}{-C/A} + \frac{y}{-C/B} = 1$ (provided $A \neq 0, B \neq 0$)

which is of the form $\frac{x}{a} + \frac{y}{b} = 1$

where x-intercept, $a = -\frac{C}{A}$ and y-intercept, $b = -\frac{C}{B}$

(iii) Reduction to the normal form : If the length of the perpendicular from the origin on $Ax + By + C = 0$ be p and if α be the angle which the perpendicular makes with the axis, then $Ax + By + C = 0$ must be same as

$x\cos\alpha + y\sin\alpha - p = 0$ ...(i)

Comparing both equations, we get

$$\cos\alpha = \pm\frac{A}{\sqrt{(A^2+B^2)}},\ \sin\alpha = \pm\frac{B}{\sqrt{(A^2+B^2)}}$$

and $p = \pm\dfrac{C}{\sqrt{(A^2+B^2)}}$

Hence, equation (i) can be written as

$$\pm\frac{A}{\sqrt{(A^2+B^2)}}x \pm \frac{B}{\sqrt{(A^2+B^2)}}y = \frac{\pm C}{\sqrt{(A^2+B^2)}}$$

which is the required form.

Sign of the equation is taken as so that p should be positive.

DISTANCE OF A POINT FROM A LINE

Length of perpendicular from a given point $P(x_1,y_1)$ on a line $Ax + By + C = 0$, is given by

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2+B^2}}.$$

Note : Length of the perpendicular from the origin on the straight line $Ax + By + C = 0$ is $\dfrac{|C|}{\sqrt{(A^2+B^2)}}$.

DISTANCE BETWEEN TWO PARALLEL LINES

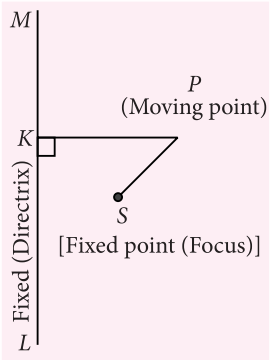
Distance between the parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by

$$d = \frac{|C_2 - C_1|}{\sqrt{A^2+B^2}}.$$

CONIC SECTIONS

DEFINITION

A conic section or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its distance from a fixed straight line. The fixed point is called focus, the fixed straight line is called directrix, and the constant ratio is called the eccentricity, which is denoted by e.



From figure, we have

$$\frac{PS}{PK} = \text{constant} = e$$

If $e = 0$, the curve is a circle.
If $e = 1$, the curve is a parabola.
If $e < 1$, the curve is an ellipse.
If $e > 1$, the curve is a hyperbola.

The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic and the point of intersection of conic with its axis is called the vertex of the conic.

CIRCLES

DEFINITION

A circle is the locus of a point which moves in a plane such that its distance from a fixed point remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

Equation of Circle

- **Standard Form :**
  Equation of circle with centre (h, k) and radius a is given by $(x - h)^2 + (y - k)^2 = a^2$
  If the centre of the circle is at origin, then $h = 0$ and $k = 0$. So, equation of circle becomes
  $x^2 + y^2 = a^2$
- **General Form :** Equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents general form of circle with centre at $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$ $(g^2 + f^2 \geq c)$.

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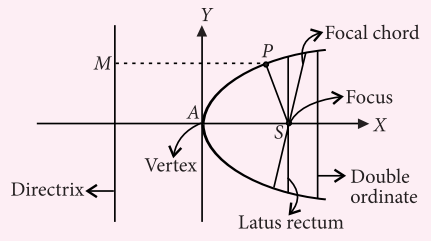
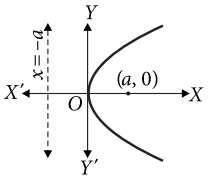
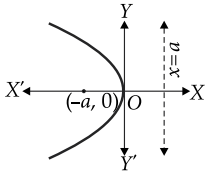
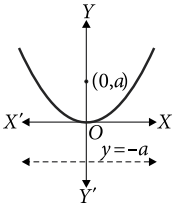
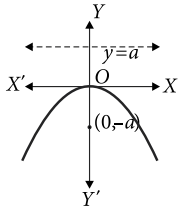
PARABOLA

DEFINITION

A Parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point.



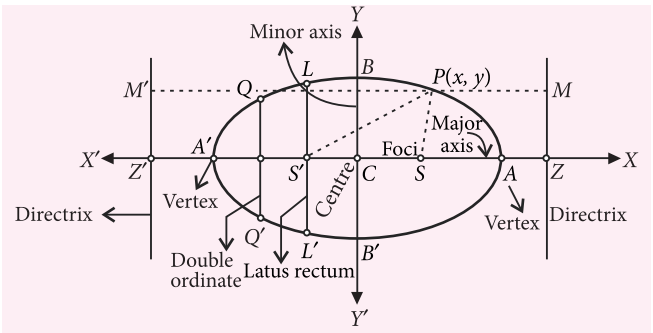
Standard equation of parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Graph				
Eccentricity	$e = 1$	$e = 1$	$e = 1$	$e = 1$
Focus	$S(a, 0)$	$S(-a, 0)$	$S(0, a)$	$S(0, -a)$
Equation of directrix	$x + a = 0$	$x - a = 0$	$y + a = 0$	$y - a = 0$
Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Vertex	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$
Extremities of latus rectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$
Equation of latus rectum	$x - a = 0$	$x + a = 0$	$y - a = 0$	$y + a = 0$

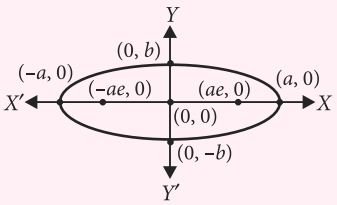
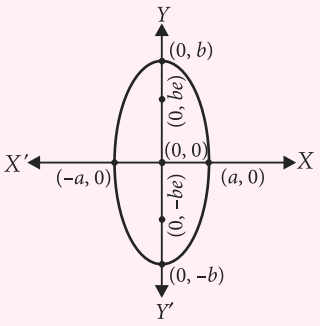
Note : Parabola is symmetric with respect to the axis of the parabola. If the equation has y^2 term, then the axis of symmetry is along the x-axis and if the equation has x^2 term, then the axis of symmetry is along the y-axis.

ELLIPSE

DEFINITION

An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is constant.



Fundamental terms	Ellipse (Horizontal ellipse)	Conjugate Ellipse (Vertical ellipse)
Standard equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a<b$
Graph		
Centre	(0, 0)	(0, 0)
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x=\pm\left(\frac{a}{e}\right)$	$y=\pm\left(\frac{b}{e}\right)$
Eccentricity	$e=\sqrt{1-\frac{b^2}{a^2}}$	$e=\sqrt{1-\frac{a^2}{b^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latus rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Focal distance or radii	$\lvert SP\rvert = (a - ex_1)$ and $\lvert S'P\rvert = (a + ex_1)$	$\lvert SP\rvert = (b - ey_1)$ and $\lvert S'P\rvert = (b + ey_1)$
Sum of focal radii = $\lvert SP\rvert + \lvert S'P\rvert$	$2a$	$2b$
Distance between foci	$2ae$	$2be$

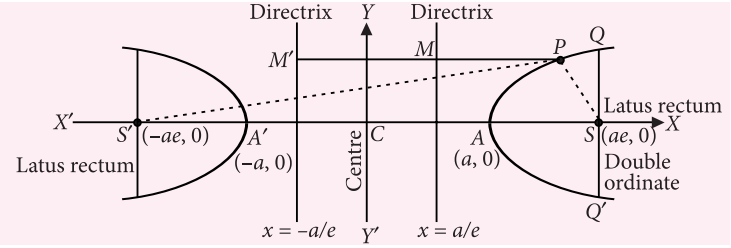
Note :

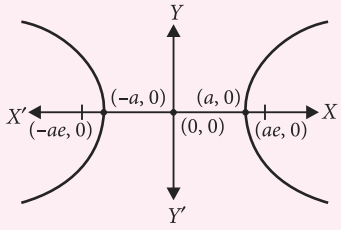
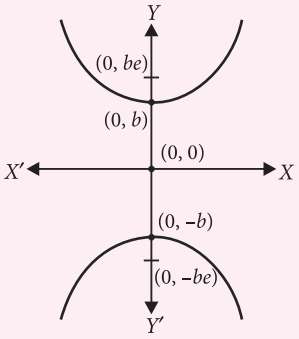
- If length of major axis is $2a$, length of minor axis is $2b$ and distance between the foci is $2c$, then $a^2 = b^2 + c^2$
- Ellipse is symmetric w.r.t. both the coordinate axes.
- The foci always lie on the major axis. *i.e,* major axes is along the x-axis if the coefficient of x^2 has the larger denominator and it is along the y-axis if the coefficient of y^2 has the larger denominator.
- If both the foci merge together with ellipse and $a = b$, the ellipse become circle.
- When distance between foci $(SS') = a$ then $b = 0$. The ellipse is reduced to the line segment SS'.

HYPERBOLA

DEFINITION

A hyperbola is the set of all points in a plane, the difference of whose distance from two fixed points is constant.



Fundamental terms	Hyperbola	Conjugate Hyperbola
Standard equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$
Graph		
Centre	(0, 0)	(0, 0)
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x=\pm\left(\frac{a}{e}\right)$	$y=\pm\left(\frac{b}{e}\right)$
Eccentricity	$e=\sqrt{1+\frac{b^2}{a^2}}$	$e=\sqrt{1+\frac{a^2}{b^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latus rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Difference of focal radii = $\lvert SP\rvert - \lvert S'P\rvert$	$2a$	$2b$
Distance between foci	$2ae$	$2be$

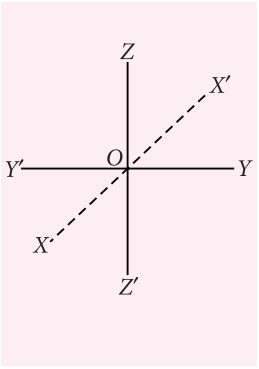
Note :

- A hyperbola in which $a = b$ is called an equilateral hyperbola.
- Hyperbola is symmetric w.r.t. both the axes.
- The foci are always on the transverse axis.

COORDINATE AXES

Let XOX', YOY', ZOZ' be the three mutually perpendiculars lines, intersecting at O.

The position of a point is located with reference to these lines. The lines XOX', YOY', ZOZ' are called the rectangular co-ordinate axes. The point of intersection O of these lines is known as origin.



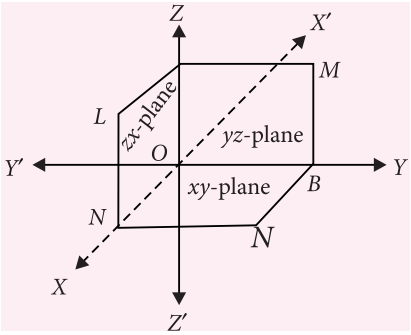
Distance along OX, OY, OZ are considered as positive, while distance in the direction OX', OY', OZ' are negative.

COORDINATE PLANES

(i) XOY is called the xy-plane.

(ii) YOZ is called the yz-plane.

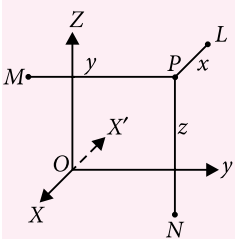
(iii) ZOX is called the zx-plane.



These planes, called the coordinate planes, divides the space into eight parts known as octants. These octants could be named as $OXYZ$, $X'OYZ$, $XOY'Z$, $X'OY'Z$, $XOYZ'$, $X'OYZ'$, $XOY'Z'$ and $X'OY'Z'$.

COORDINATES OF A POINT IN SPACE

Let P be any point in space. Draw PL, PM, PN perpendicular to the yz, zx and xy planes respectively then



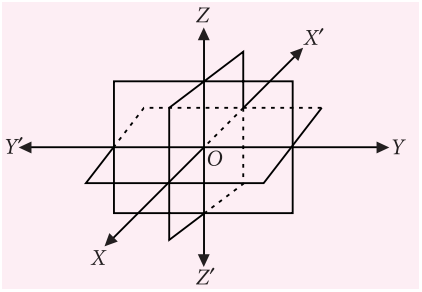
(i) LP is called the x-coordinate of P.

(ii) MP is called the y-coordinate of P.

(iii) NP is called the z-coordinate of P.

These three, taken together, are known as coordinates of P. Thus the coordinates of any point in space are the perpendicular distances of the point from the yz, zx and xy planes respectively.

Signs of Coordinates in Eight Octants :



The octants $XOYZ$, $X'OYZ$, $X'OY'Z$, $XOY'Z$, $XOYZ'$, $X'OYZ'$, $X'OY'Z'$ and $XOY'Z'$ are denoted by I, II, III, ..., VIII respectively.

	I	II	III	IV	V	VI	VII	VIII
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

DISTANCE BETWEEN TWO POINTS

The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

SECTION FORMULA

Let M be a point which divides the line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $m : n$, then

Types of division	Coordinates of M
Internal	$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$
External	$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$
Mid-point	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

COORDINATES OF CENTROID OF A TRIANGLE

The coordinates of the centroid of a triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Very Short Answer Type

1. What is the inclination of a line whose slope is $-\sqrt{3}$.
2. Find the co-ordinates of the focus and equation of directrix of the parabola $x^2 = -16y$
3. Find the value(s) of k for which the line $(k-3)x + (k^2-4)y + (k-1)(k-6) = 0$ is parallel to the x-axis,
4. Find the equation of a circle with centre $(3, -2)$ and radius 5.
5. Find the coordinates of the point which divides the join of $P(2, -1, 4)$ and $Q(4, 3, 2)$ in the ratio 2 : 3 externally.

Short Answer Type

6. If $A(-2, 1)$, $B(2, 3)$ and $C(-2, -4)$ are three points, find the angle between BA and BC.
7. (i) Find the equation of a line that has y-intercept 4 and is perpendicular to the line joining $(2, -3)$ and $(4, 2)$.
   (ii) Find the equation of the line which cuts off equal and positive intercepts from the axes and passes through the point (α, β).
8. Reduce the lines $3x - 4y + 4 = 0$ and $4x - 3y + 12 = 0$ to the normal form and hence determine which line is nearer to the origin.
9. Find the equation of the hyperbola whose foci are $(0, \pm\sqrt{10})$ and which passes through the point $(2, 3)$.
10. The centroid of a triangle ABC is at the point $(1, 1, 1)$. If the coordinates of A and B are $(3, -5, 7)$ and $(-1, 7, -6)$, respectively, find the coordinates of the point C.

Long Answer Type

11. The equation of a straight line which passes through the point $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to $x \sec\theta + y \operatorname{cosec}\theta = a$ is
12. Find the length of axes of the ellipse whose eccentricity is 4/5 and whose foci coincide with those of the hyperbola $9x^2 - 16y^2 + 144 = 0$.
13. Find the equation of the circle passing through the points $(3, 7)$, $(5, 5)$ and having its centre on the line $x - 4y = 1$.
14. (i) If the point $(h, 0)$, (a, b) and $(0, k)$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.
    (ii) Using slopes, show that the points $(5, 1)$, $(1, -1)$ and $(11, 4)$ are collinear.
15. Find the perpendicular distance of the line joining the points $A(\cos\theta, \sin\theta)$ and $B(\cos\phi, \sin\phi)$ from the origin.

SOLUTIONS

1. We have, $m = -\sqrt{3} \Rightarrow \tan\theta = -\sqrt{3}$
   $\Rightarrow \tan\theta = \tan(180 - 60°) \Rightarrow \theta = 120°$
2. We have, equation of parabola, $x^2 = -16y$
   Comparing with $x^2 = -4ay$, we get $a = 4$
   Coordinates of focus are $(0, -4)$.
   Equation of directrix is $y - 4 = 0$
3. Given equation of the line is
   $(k-3)x + (k^2-4)y + (k-1)(k-6) = 0$
   The given line will be parallel to the x-axis, when the coefficient of y is zero,
   i.e., when $k^2 - 4 = 0$
   $\Rightarrow k = \pm 2$.
4. We know that the equation of a circle with centre $C(h, k)$ and radius r is given by
   $(r-h)^2 + (y-k)^2 = r^2$.
   Here, $h = 3$, $k = -2$ and $r = 5$.
   $\therefore$ The required equation of the circle is
   $(x-3)^2 + (y+2)^2 = 5^2$
   $\Rightarrow x^2 + y^2 - 6x + 4y - 12 = 0$.
5. We have, $P(2, -1, 4)$, $Q(4, 3, 2)$ and $m_1 : m_2 = 2 : 3$
   Let $R(x, y, z)$ be the required point. Then,
   $x = \frac{(2\times4)-(3\times2)}{2-3}$, $y = \frac{(2\times3)-(3\times-1)}{2-3}$, $z = \frac{(2\times2)-(3\times4)}{2-3}$
   $\Rightarrow x = -2, y = -9, z = 8$.
   So, required point is $R(-2, -9, 8)$.

6. We have, $A(-2, 1)$, $B(2, 3)$ and $C(-2, -4)$
Let m_1 and m_2 be the slopes of BA and BC respectively. Then,

$$m_1 = \frac{3-1}{2-(-2)} = \frac{2}{4} = \frac{1}{2}, \text{ and } m_2 = \frac{-4-3}{-2-2} = \frac{7}{4}$$

Let θ be the angle between BA and BC. Then,

$$\tan\theta = \left|\frac{m_2 - m_1}{1+m_1m_2}\right| = \left|\frac{\frac{7}{4}-\frac{1}{2}}{1+\frac{7}{4}\times\frac{1}{2}}\right| = \left|\frac{\frac{5}{4}}{\frac{15}{8}}\right| = \frac{2}{3}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{2}{3}\right)$$

7. (i) Let m be the slope of the required line. Since, the required line is perpendicular to the line joining $A(2, -3)$ and $B(4, 2)$.

$\therefore$ $m \times$ slope of $AB = -1$

$$\Rightarrow \quad m \times \frac{2+3}{4-2} = -1 \quad \Rightarrow \quad m = -\frac{2}{5}$$

Also, the required line has y-intercept 4. So, $c = 4$
Substituting the value of m and c in $y = mx + c$, we get

$y = -\frac{2}{5}x + 4 \Rightarrow 2x + 5y - 20 = 0$ which is the required equation of line

(ii) Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ which cuts off intercepts a and b with the coordinate axes. It is given that $a = b$. Therefore the equation of the line is

$$\frac{x}{a} + \frac{y}{a} = 1 \quad \Rightarrow \quad x + y = a \qquad \text{... (i)}$$

But, it passes through (α, β). So, $\alpha + \beta = a$.
Putting the value of a in (i), we get
$x + y = \alpha + \beta$ which the required equation of line

8. We have,

$3x - 4y + 4 = 0$

$$\Rightarrow \quad -3x + 4y = 4$$

$$\Rightarrow \quad -\frac{3x}{\sqrt{(-3)^2+4^2}} + \frac{4y}{\sqrt{(-3)^2+4^2}} = \frac{4}{\sqrt{(-3)^2+4^2}}$$

$$\Rightarrow \quad -\frac{3}{5}x + \frac{4}{5}y = \frac{4}{5}$$

This is the normal form of $3x - 4y + 4 = 0$ and the length of the perpendicular from the origin to it is given by $p_1 = 4/5$
Now, $4x - 3y + 12 = 0$

$$\Rightarrow \quad -4x + 3y = 12$$

$$\Rightarrow \quad -\frac{4x}{\sqrt{(-4)^2+3^2}} + \frac{3y}{\sqrt{(-4)^2+3^2}} = \frac{12}{\sqrt{(-4)^2+3^2}}$$

$$\Rightarrow \quad -\frac{4}{5}x + \frac{3}{5}y = \frac{12}{5}$$

This is the normal form of $4x - 3y + 12 = 0$ and the length of the perpendicular from the origin to it is given by $p_2 = \frac{12}{5}$.

Clearly, $p_2 > p_1$. Therefore, the line $3x - 4y + 4 = 0$ is nearer to the origin.

9. The foci of hyperbola are at $(0, \pm\sqrt{10})$. These are on the y-axis.
Clearly, centre of the hyperbola will be $(0, 0)$. Let equation of the hyperbola be

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \qquad \text{...(i)}$$

where $a, b > 0$ and $b^2 = a^2(e^2 - 1)$
The foci of this hyperbola are at $(0, \pm ae)$
$\therefore$ $ae = \sqrt{10}$.
Now, $b^2 = a^2(e^2 - 1) = a^2e^2 - a^2 = 10 - a^2$.
From (i), the equation of hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{10-a^2} = 1 \qquad \text{...(ii)}$$

Since (2, 3) lies on hyperbola (ii),

$$\therefore \quad \frac{9}{a^2} - \frac{4}{10-a^2} - = 1$$

$\Rightarrow 90 - 9a^2 - 4a^2 = 10a^2 - a^4 \Rightarrow a^4 - 23a^2 + 90 = 0$
$\Rightarrow a^2 = 18, 5$
$a^2 = 18 \Rightarrow b^2 = 10 - 18 = -8$ not possible.
$a^2 = 5 \Rightarrow b^2 = 10 - 5 = 5$

$\therefore$ From (i), the equation of hyperbola is $\frac{y^2}{5} - \frac{x^2}{5} = 1$
or $y^2 - x^2 = 5$.

10. Let the coordinates of C be (x, y, z). Coordinates of the centroid $G = (1, 1, 1)$. Then

$$\frac{x+3-1}{3} = 1, \ \frac{y-5+7}{3} = 1, \ \frac{z+7-6}{3} = 1$$

$\Rightarrow \quad x = 1, y = 1, z = 2.$
Hence, coordinates of C are $(1, 1, 2)$.

11. We have $x\sec\theta + y\,\text{cosec}\theta = a$

$$\Rightarrow \quad y\,\text{cosec}\theta = a - x\sec\theta$$

$$\Rightarrow \quad y = \frac{-\sec\theta}{\text{cosec}\,\theta}x + \frac{a}{\text{cosec}\,\theta}$$

$\Rightarrow y = (-\tan\theta)x + a\sin\theta$

$\Rightarrow$ slope = $-\tan\theta$

Slope of line $\perp$ to $x\sec\theta + y\,\text{cosec}\theta = a$ is $\dfrac{1}{\tan\theta} = \cot\theta$

$\therefore$ Equation of line through $(a\cos^3\theta, a\sin^3\theta)$ and slope $\cot\theta$ is $y - a\sin^3\theta = \cot\theta\,(x - a\cos^3\theta)$

$$\Rightarrow \quad y - x\frac{\cos\theta}{\sin\theta} = a\sin^3\theta - a\cos^3\theta\cot\theta$$

$$\Rightarrow \quad x\cos\theta - y\sin\theta = \sin\theta\left[-a\sin^3\theta + a\frac{\cos^4\theta}{\sin\theta}\right]$$

$\Rightarrow$ $x\cos\theta - y\sin\theta = a[-\sin^4\theta + \cos^4\theta]$

$= a(\sin^2\theta + \cos^2\theta)(-\sin^2\theta + \cos^2\theta)$

$= a(\cos^2\theta - \sin^2\theta) = a\cos 2\theta$

$\Rightarrow$ $x\cos\theta - y\sin\theta = a\cos 2\theta$

12. The equation of the hyperbola can be written as

$\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$ or $-\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ Here $a = 3$, $b = 4$

The eccentricity of the hyperbola is given by

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$$

Since transverse axis of the hyperbola is along y-axis, its foci are $(0, \pm ae)$,

i.e., $(0, \pm (5/3) \times 3)$ *i.e.*, $(0, \pm 5)$.

The ellipse whose foci are $S(0, 5)$ and $S'(0, -5)$ has its major axis along y-axis.

Its centre is $(0, 0)$ [mid-point of $S'S$]

Let $2a$ and $2b$ be the lengths of major and minor axes of the ellipse; then its equation will be

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad \ldots (1)$$

Since $2ae = S'S = 10 \Rightarrow a\cdot\dfrac{4}{5} = 5 \Rightarrow a = \dfrac{25}{4}$

Now, $b^2 = a^2(1 - e^2)$

$$\therefore \quad b^2 = \frac{625}{16}\left(1 - \frac{16}{25}\right) = \frac{225}{16}$$

Hence, equation of the required ellipse is

$$\frac{16}{225}x^2 + \frac{16}{625}y^2 = 1$$

Length of major axis of the ellipse $= 2a = \dfrac{25}{2}$ and

Length of minor axis $= 2b = \dfrac{15}{2}$.

13. Let the equation of the required circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ... (i)

It passes through the point (3, 7)

$\therefore$ $9 + 49 + 6g + 14f + c = 0$

or $6g + 14f + c + 58 = 0$ ... (ii)

Also point (5, 5) lies on (i)

$\therefore$ $25 + 25 + 10g + 10f + c = 0$

or $10g + 10f + c + 50 = 0$ ... (iii)

Centre of (i), *i.e.*, $(-g, -f)$ lies on line $x - 4y = 1$

$\therefore$ $-g + 4f = 1$ or $g - 4f + 1 = 0$ ... (iv)

Subtracting (iii) from(ii), we get

$-4g + 4f + 8 = 0$ or $-g + f + 2 = 0$ ... (v)

Adding (iv) and (v), we get

$-3f + 3 = 0 \Rightarrow f = 1$

Putting $f = 1$ in (iv), we have

$g - 4 + 1 = 0 \Rightarrow g = 3$

Putting $g = 3$, $f = 1$ in (ii), we get

$6(3) + 14(1) + c + 58 = 0$

or $c + 18 + 14 + 58 = 0$

or $c + 90 = 0 \Rightarrow c = -90$

Substituting the values of $g = 3$, $f = 1$ and $c = -90$ in (i), we get

$x^2 + y^2 + 6x + 2y - 90 = 0$

which is the required equation of the circle.

14. (i) Let $A(h, 0)$, $B(a, b)$ and $C(0, k)$ be the given points.

Since the given point A, B, C are collinear, we have slope of AB = slope of BC

$$\therefore \quad \frac{b-0}{a-h} = \frac{k-b}{0-a} \Leftrightarrow \frac{b}{(a-h)} = \frac{(b-k)}{a}$$

$\Leftrightarrow$ $ab = (a - h)(b - k)$

$\Leftrightarrow$ $ak + hb = hk$

$\Leftrightarrow$ $\dfrac{a}{h} + \dfrac{b}{k} = 1$ [on dividing both sides by hk]

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(ii) Let $A(5, 1)$, $B(1, -1)$ and $C(11, 4)$ be the given points. Then,

Slope of $AB = \frac{(-1-1)}{(1-5)} = \frac{-2}{-4} = \frac{1}{2}$

and slope of $BC = \frac{4-(-1)}{11-1} = \frac{5}{10} = \frac{1}{2}$

$\therefore$ Slope of AB= Slope of BC

$\Rightarrow$ $AB \parallel BC$ and have a point B in common.

$\Rightarrow$ A, B, C are collinear.

Hence, the given points are collinear.

15. We know that the equation of the line AB is given by

$$\frac{y-\sin\theta}{x-\cos\theta} = \frac{\sin\phi-\sin\theta}{\cos\phi-\cos\theta}$$

$$\Rightarrow \frac{y-\sin\theta}{x-\cos\theta} = \frac{2\cos\left(\frac{\phi+\theta}{2}\right)\sin\left(\frac{\phi-\theta}{2}\right)}{-2\sin\left(\frac{\phi+\theta}{2}\right)\sin\left(\frac{\phi-\theta}{2}\right)}$$

$$\Rightarrow \frac{y-\sin\theta}{x-\cos\theta} = \frac{\cos\left(\frac{\phi+\theta}{2}\right)}{-\sin\left(\frac{\phi+\theta}{2}\right)}$$

$$\Rightarrow x\cos\left(\frac{\phi+\theta}{2}\right)+y\sin\left(\frac{\phi+\theta}{2}\right) - \left\{\cos\theta.\cos\left(\frac{\phi+\theta}{2}\right)+\sin\theta\cdot\sin\left(\frac{\phi+\theta}{2}\right)\right\}=0$$

$$\Rightarrow x\cos\left(\frac{\phi+\theta}{2}\right)+y\sin\left(\frac{\phi+\theta}{2}\right)-\cos\left(\frac{\phi+\theta}{2}-\theta\right)=0$$

$$\Rightarrow x\cos\left(\frac{\phi+\theta}{2}\right)+y\sin\left(\frac{\phi+\theta}{2}\right)-\cos\left(\frac{\phi-\theta}{2}\right)=0 \quad \ldots (i)$$

Let d be the perpendicular distance from the origin to line (i). Then,

$$d = \frac{0\times\cos\left(\frac{\phi+\theta}{2}\right)+0\times\sin\left(\frac{\phi+\theta}{2}\right)-\cos\left(\frac{\phi-\theta}{2}\right)}{\sqrt{\cos^2\left(\frac{\phi+\theta}{2}\right)+\sin^2\left(\frac{\phi+\theta}{2}\right)}}$$

$$= \left|\cos\left(\frac{\phi-\theta}{2}\right)\right|$$

Hence, the required distance is $\left|\cos\left(\frac{\phi-\theta}{2}\right)\right|$.

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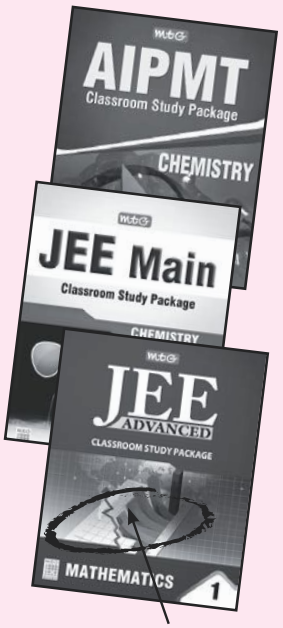

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CONCEPT BOOSTERS

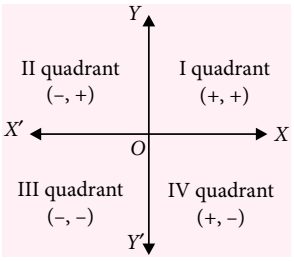
Class XI

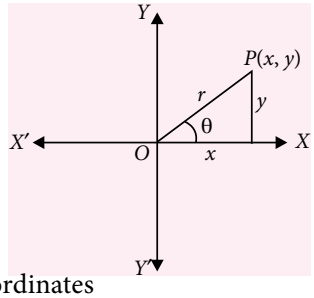
STRAIGHT LINES

*ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

CARTESIAN & POLAR COORDINATES





$P(x, y)$ - Cartesian coordinates

$P(r, \theta)$ - Polar coordinates

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}$$

DISTANCE FORMULA, SECTION FORMULA & AREA

- **Distance formula :** Distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- **Section formula :** The coordinates of a point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) internally, in the ratio $m : n$ are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

and externally in the ratio $m : n$ are

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

- **Mid-Point formula :** In particular, if $m = n$, the coordinates of the mid-point of the line segment joining the points (x_1, y_1) and (x_2, y_2) is

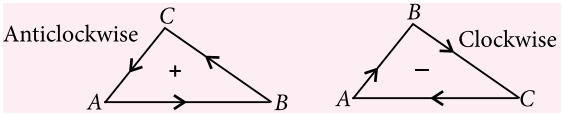
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

- **Area of a Triangle :** Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively be the coordinates of the vertices A, B, C of a ΔABC. Then the area of triangle ΔABC, is

$$\frac{1}{2}\left|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\right|$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

While using formulas, order of the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) has not been taken into account. If we plot the points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then the area of the triangle as obtained by using formulas will be positive or negative as the points A, B, C are in anticlockwise or clockwise directions.



So, while finding the area of triangle ABC, we take modulus.

Remark :

In case of polygon with vertices (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) in order, then area of polygon is given by

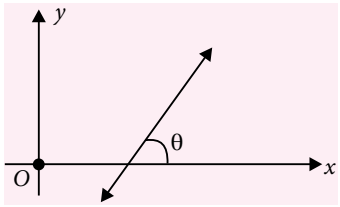
$$\frac{1}{2}|(x_1y_2 - y_1x_2) + (x_2y_3 - y_2x_3) + ... + (x_{n-1}y_n - y_{n-1}x_n) + (x_ny_1 - y_nx_1)|$$

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

SLOPE & EQUATION OF LINE

- If θ is the inclination of a line L, $(0 \le \theta < \pi)$ then tanθ is called the slope or gradient of the line L.

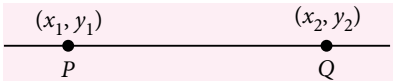
The slope of a line whose inclination is 90° is not defined.
The slope of a line is denoted by m.
Thus, $m = \tan\theta$, $\theta \neq 90°$



It may be observed that the slope of x-axis is zero and slope of y-axis is not defined.

- **Slope of a line when coordinates of any two points on the line are given :**

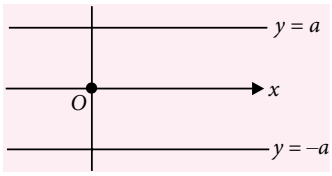
$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta = \frac{dy}{dx}$$



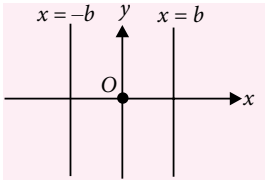
- **Conditions for parallelism and perpendicularity of lines in terms of their slopes :**
If lines $L_1 \parallel L_2$, then $m_1 = m_2$
If $L_1 \perp L_2$, then $m_1 \cdot m_2 = -1$

VARIOUS FORMS OF THE EQUATION OF A LINE:

- **Line Parallel to x-axis**



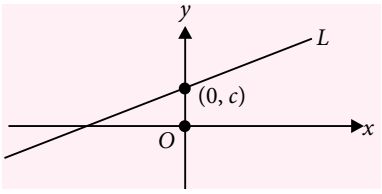
- **Line Parallel to y-axis**



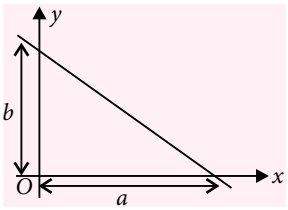
- **Point slope form**
Equation of line L having slope m and passing through (x_1, y_1) is
$y - y_1 = m(x - x_1)$
- **Two point form :** Equation of line L passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad \text{or} \quad y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$$

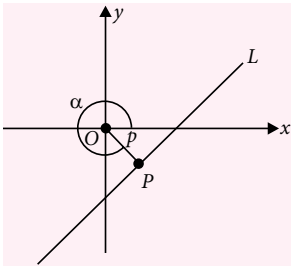
- **Slope-intercept form**
If line L having slope m and intercept on y-axis is c, then equation of line L is $y = mx + c$



- **Intercept form**
Suppose a line L makes intercept 'a' on x-axis and 'b' on y-axis. Then its equation is $\frac{x}{a} + \frac{y}{b} = 1$



- **Normal form**
Let length of perpendicular from origin to line L is $OP = p$ and α is the angle which is made by OP with +ve x-axis.
The equation of line L is $x\cos\alpha + y\sin\alpha = p$



- **General Equation of a Line**
$Ax + By + C = 0$

DIFFERENT FORMS OF $Ax + By + C = 0$

- **Slope Intercept Form**
If $B \neq 0$, then $Ax + By + C = 0$ ....(i)
can be written as

$$y = -\frac{A}{B}x - \frac{C}{B} \quad \text{....(ii)}$$

Compare (ii) with $y = mx + c$, we get

$$\text{Slope of line} = m = -\frac{A}{B}$$

$$\text{Intercept on } y\text{-axis} = c = \frac{-C}{B}$$

- **Intercept form**
If $C \neq 0$, then $Ax + By + C = 0$ ....(i)
can be written as

$$\frac{x}{\left(\frac{-C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1 \quad \text{...(ii)}$$

Compare (ii) with $\frac{x}{a} + \frac{y}{b} = 1$, we get ....(iii)

$$a = \text{Intercept on } x\text{-axis} = -\frac{C}{A}$$

$$b = \text{Intercept on } y\text{-axis} = -\frac{C}{B}$$

- **Normal form**
Let $Ax + By + C = 0$ ....(i)
can be written as $Ax + By = -C$ ....(ii)
Compare (ii) with $x\cos\alpha + y\sin\alpha = p$

so $\frac{A}{\cos\alpha}=\frac{B}{\sin\alpha}=\frac{-C}{p}$

$\Rightarrow \cos\alpha=\frac{-pA}{C}$ and $\sin\alpha=\frac{-pB}{C}$

Using $\sin^2\alpha + \cos^2\alpha = 1$

$\Rightarrow \left(\frac{-pA}{C}\right)^2+\left(\frac{-pB}{C}\right)^2=1$

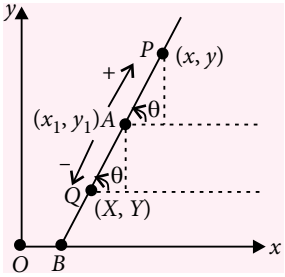
$\Rightarrow p^2=\frac{C^2}{A^2+B^2}\Rightarrow p=\pm\frac{C}{\sqrt{A^2+B^2}}$

Proper choice of signs is made so that p should be +ve.

So, $\cos\alpha=\pm\frac{A}{\sqrt{A^2+B^2}}$ and $\sin\alpha=\pm\frac{B}{\sqrt{A^2+B^2}}$

PARAMETRIC EQUATIONS OF A STRAIGHT LINE

In figure given below, let BAP be a straight line through a given point $A\ (x_1, y_1)$, the angle of slope being θ. The positive direction of the line is in the sense BAP. (Direction of increasing ordinate is called the positive direction of the line).



For the points $P\ (x, y)$ and $Q\ (X, Y)$ shown in the figure, AP is regarded as a positive vector and AQ as a negative vector, as indicated by the arrows.

From the general definitions of cosθ and sinθ, we have

$\cos\theta=\frac{x-x_1}{AP}$, $\sin\theta=\frac{y-y_1}{AP}$

or $x - x_1 = AP\cos\theta$, $y - y_1 = AP\sin\theta$

or $\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=\pm r$

ANGLE BETWEEN TWO STRAIGHT LINES AND COLLINEAR POINTS

- **Angle between two lines**

The acute angle between lines L_1 and L_2 with slopes m_1 and m_2 respectively, is given by

$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$, as $m_1m_2 \neq -1$

Thus obtuse angle $\phi = 180° - \theta$

- **Collinearity of three points**

If A, B, C are collinear, then

- $m_{AB} = m_{BC} = m_{AC}$
- $|AB| + |BC| = |AC|$ or $|BC| + |CA| = |BA|$ or $|CA| + |AB| = |CB|$

LENGTH OF PERPENDICULAR AND REFLECTION

- Length of perpendicular from (x_1, y_1) to a straight line $ax + by + c = 0$ is given by

$$\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$

If two lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ are parallel, then $d=\left|\frac{c_1-c_2}{\sqrt{a^2+b^2}}\right|$

- **Reflection of a point about a line**

Let the image of a point (x_1, y_1) about the line $ax + by + c = 0$ is (x_2, y_2), then

$$\frac{x_2-x_1}{a}=\frac{y_2-y_1}{b}=-2\left(\frac{ax_1+by_1+c}{a^2+b^2}\right)$$

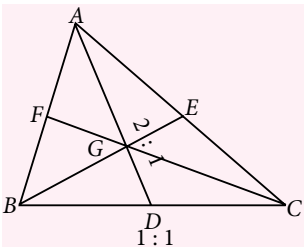
and let the foot of perpendicular from a point (x_1, y_1) on the line $ax + by + c = 0$ is (x_2, y_2) then

$$\frac{x_2-x_1}{a}=\frac{y_2-y_1}{b}=-\left(\frac{ax_1+by_1+c}{a^2+b^2}\right)$$

CENTRES CONNECTED WITH A TRIANGLE

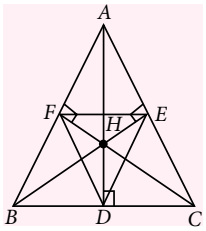
(w.r.t. ΔABC, where $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$, $C \equiv (x_3, y_3)$, $BC = a$, $CA = b$ & $AB = c$).

- **Centroid :** The point of concurrency of the medians of a triangle is called the centroid of the triangle. The centroid of a triangle divides each median in the ratio 2 : 1. The coordinates of centroid are given by $G\equiv\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$



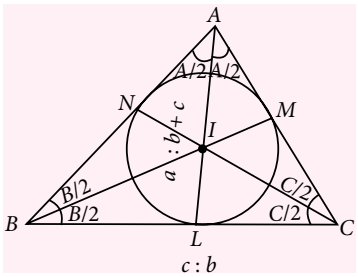
- **Orthocentre :** The point of concurrency of the altitudes of a triangle is called the orthocentre of the triangle. The coordinates of the orthocentre are given by

$$H\equiv\left(\frac{x_1\tan A+x_2\tan B+x_3\tan C}{\tan A+\tan B+\tan C},\ \frac{y_1\tan A+y_2\tan B+y_3\tan C}{\tan A+\tan B+\tan C}\right)$$



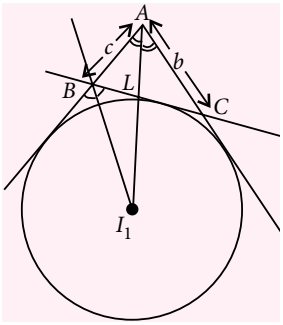
The triangle formed by joining the feet of altitudes in a triangle is called the orthic triangle. Here ΔDEF is the orthic triangle of ΔABC.

- **Incentre :** The point of concurrency of the internal bisectors of the angles of a triangle is called the incentre of the triangle. The coordinates of the incentre are given by



$$I \equiv \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right)$$

- **Excentre :** Coordinate of excentre opposite to $\angle A$ is given by $\frac{BL}{LC} = \frac{c}{b}$,



Also $\frac{AI_1}{I_1L} = -\frac{b+c}{a}$

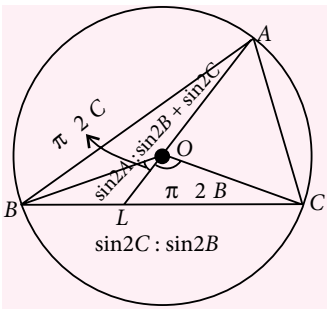
$$I_1 \equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c} \right)$$

and similarly for excentres (I_2 & I_3) opposite to AB and AC are given by

$$I_2 \equiv \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c} \right)$$

$$I_3 \equiv \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c} \right)$$

- **Circumcentre :** The point of concurrency of the perpendicular bisectors of the sides of a triangle is called circumcentre of the triangle. The coordinates of the circumcentre are given by



$$O \equiv \left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} \right)$$

Remarks :

- Circumcentre O, centroid G and orthocentre H of a ΔABC are collinear. G divides OH in the ratio 1 : 2, *i.e.* $OG : GH = 1 : 2$
- In an isosceles triangle centroid, orthocentre, incentre and circumcentre lie on the same line and in an equilateral triangle all these four points coincide.

POSITION OF A GIVEN POINT RELATIVE TO A GIVEN LINE & CONCURRENCY

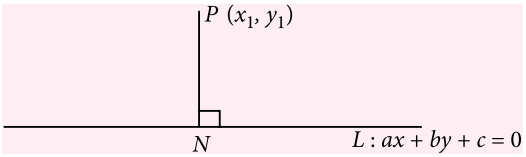
The figure shows a point $P(x_1, y_1)$ lying above a given line. If an ordinate is dropped from P to meet the line L at N, then the x-coordinate of N will be x_1.

Putting $x = x_1$ in the equation $ax + by + c = 0$ gives y-coordinate of $N = -\frac{(ax_1 + c)}{b}$

If $P(x_1, y_1)$ lies above the line, then we have

$$y_1 > -\frac{(ax_1 + c)}{b} \text{ i.e. } y_1 + \frac{(ax_1 + c)}{b} > 0$$

$$\text{i.e. } \frac{(ax_1 + by_1 + c)}{b} > 0 \quad \text{i.e. } \frac{L(x_1, y_1)}{b} > 0 \quad \ldots. \text{ (i)}$$



Hence, if $P(x_1, y_1)$ satisfies equation (i), it would mean that P lies above the line $ax + by + c = 0$, and if $\frac{L(x_1, y_1)}{b} < 0$, it would mean that P lies below the line $ax + by + c = 0$.

Remark :

If $(ax_1 + by_1 + c)$ and $(ax_2 + by_2 + c)$ have same signs, it implies that (x_1, y_1) and (x_2, y_2) both lie on the same side of the line $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

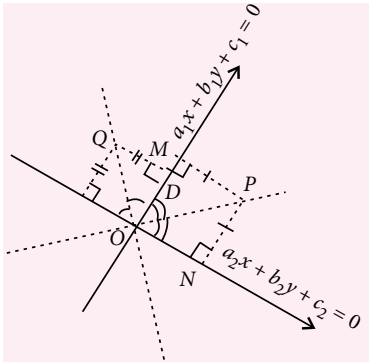
- **Concurrency of straight lines :** The condition for three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ to be concurrent is

(i) $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

(ii) There exist three constants l, m, n (not all zero at the same time) such that $lL_1 + mL_2 + nL_3 = 0$, where $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$ are the three given straight lines.

(iii) The three lines are concurrent if any one of the lines passes through the point of intersection of the other two lines.

ANGLE BISECTORS OF TWO STRAIGHT LINES



Angle bisector is the locus of a point which moves in such a way so that its distance from two intersecting lines remains same.

The equations of the two bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

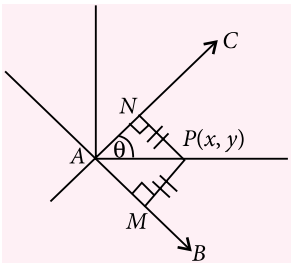
$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

If the two given lines are not perpendicular *i.e.*, $a_1a_2 + b_1b_2 \neq 0$ and not parallel *i.e.*, $a_1b_2 \neq a_2b_1$, then one of these equations is the equation of the bisector of the acute angle between two given lines and the other is the obtuse angle between two given lines.

Remark :

Whether both given lines are perpendicular or not but the angular bisectors of these lines will always be mutually perpendicular.

- **The bisectors of the acute and the obtuse angles :**



Take one of the lines and let its slope be m_1 and take one of the bisectors and let its slope be m_2. If θ be the acute angle between them, then find $\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$

If $\tan\theta > 1$ then the bisector taken is the bisector of the obtuse angle and the other one will be the bisector of the acute angle.

If $0 < \tan\theta < 1$ then the bisector taken is the bisector of the acute angle and the other one will be the bisector of the obtuse angle.

If two lines are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

will represent the equation of the bisector of the acute or obtuse angle between the lines according as $a_1a_2 + b_1b_2$ is negative or positive.

- **The equation of the bisector of the angle which contains a given point :**

The equation of the bisector of the angle between the two lines containing the point (a, b) is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

or $$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

according as $a_1a + b_1b + c_1$ and $a_2a + b_2b + c_2$ are of the same signs or of opposite signs.

For example, the equation of the bisector of the angle containing the origin is given by

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = +\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

for same sign of c_1 and c_2 (for opposite sign take –ve sign in place of +ve sign)

Remark :

(i) If $c_1c_2(a_1a_2 + b_1b_2) < 0$, then the origin will lie in the acute angle and if $c_1c_2(a_1a_2 + b_1b_2) > 0$, then the origin will lie in the obtuse angle.

(ii) Equation of straight lines passing through $P(x_1, y_1)$ and equally inclined with the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisectors between these two lines and passing through the point P.

LOCUS

When a point moves in a plane under certain geometrical conditions, the point traces out a path. This path of the moving point is called its locus.

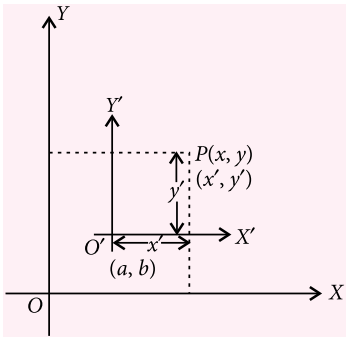
- **Equation of locus :** The equation to a locus is the relation which exists between the coordinates of

any point on the path, and which holds for no other point except those lying on the path. In other words equation to a curve (or locus) is merely the equation connecting the x and the y coordinates of every point on the curve.

- **Procedure for finding the equation of the locus of a point :**
  (i) If we are finding the equation of the locus of a point P, assign coordinates (h, k) or (x_1, y_1) to P.
  (ii) Express the given conditions in terms of the known quantities to facilitate calculations. We sometimes include some unknown quantities known as parameters.
  (iii) Eliminate the parameters. So that the eliminant contains only h, k and known quantities. If h and k coordinates of the moving point are obtained in terms of a third variable 't' called the parameter, eliminate 't' to obtain the relation in h and k and simplify this relation.
  (iv) Replace h by x, and k by y, in the eliminant. The resulting equation would be the equation of the locus of P.

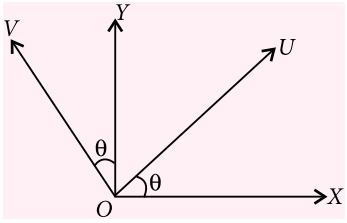
TRANSFORMATION OF AXES

- **Changes of Axes (Shifting of origin without rotation of axes) :**



Let $P \equiv (x, y)$ with respect to axes OX and OY.
Let $O' \equiv (a, b)$ with respect to axes OX and OY and let $P \equiv (x', y')$ with respect to axes $O'X'$ and $O'Y'$ where OX and $O'X'$ are parallel and OY and $O'Y'$ are parallel.
then $x = x' + a$, $y = y' + b$ or $x' = x - a$, $y' = y - b$
Thus if origin is shifted to point (a, b) without rotation of axes, then new equation of curve can be obtained by putting $x + a$ in place of x and $y + b$ in place of y.

- **Rotation of the axes (To change the direction of the axes of coordinates, without changing the origin, both systems of co ordinates being rectangular) :**
Let OX, OY be given rectangular axes with respect to which the coordinates of a point P are (x, y). Suppose that OU, OV are the two perpendicular lines obtained by rotating OX, OY respectively through an angle θ in the counter-clockwise sense. We take OU, OV as a new pair of coordinate axes, with respect to which the coordinates of P are (x', y'), then



$x = x'\cos\theta - y'\sin\theta$
$y = x'\sin\theta + y'\cos\theta$

PAIR OF STRAIGHT LINES

- The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

$\Rightarrow \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

The homogeneous second degree equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines through the origin.
If lines through the origin whose joint equation is $ax^2 + 2hxy + by^2 = 0$, are $y = m_1x$ and $y = m_2x$, then $y^2 - (m_1 + m_2)xy + m_1m_2x^2 = 0$ and $y^2 + \frac{2h}{b}xy + \frac{a}{b}x^2 = 0$ are identical. If θ is the angle between the two lines, then

$$\tan\theta = \pm\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1+m_1m_2} = \pm\frac{2\sqrt{h^2-ab}}{a+b}$$

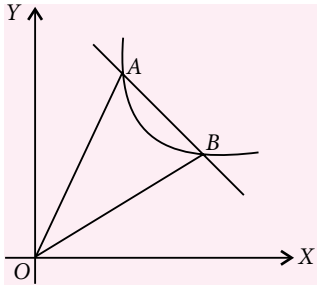
Note :
(i) (a) $a + b = 0 \Rightarrow$ lines are perpendicular.
(b) $h^2 > ab \Rightarrow$ lines are real & distinct.
(c) $h^2 = ab \Rightarrow$ lines are coincident.
(d) $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection *i.e.* (0, 0).
(ii) If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then

$$m_1 + m_2 = -\frac{2h}{b} \quad \& \quad m_1m_2 = \frac{a}{b}.$$

(iii) The equation to the straight lines bisecting the angle between the straight lines $ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

(iv) A homogeneous equation of degree n represents n straight lines (in general) passing through origin.

- **Joint equation of pair of lines joining the origin and the points of intersection of a line and a curve**



If the line $lx + my + n = 0$, $((n \neq 0)$ *i.e.* the line not passing through origin) cuts the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at two points A and B, then the joint equation of straight lines passing through A and B and the origin is given by homogenizing the equation of the curve by the equation of the line *i.e.*

$$ax^2 + 2hxy + by^2 + (2gx + 2fy)\left(\frac{lx+my}{-n}\right) + c\left(\frac{lx+my}{-n}\right)^2 = 0$$

is the equation of the lines OA and OB.

PROBLEMS

SECTION-I

Single Correct Answer Type

1. The line $x + y = 1$ meets x-axis at A and y-axis at B. P is the mid-point of AB. P_1 is the foot of the perpendicular from P to OA; M_1 is that from P_1 to OP; P_2 is that from M_1 to OA; M_2 is that from P_2 to OP; P_3 is that from M_2 to OA and so on. If P_n denotes the n^{th} foot of the perpendicular on OA from M_{n-1}, then $OP_n =$

(a) $\frac{1}{2}$ (b) $\frac{1}{2^n}$ (c) $\frac{1}{2^{n/2}}$ (d) $\frac{1}{\sqrt{2}}$

2. The orthocentre of the triangle formed by the lines $x + y = 1$, $2x + 3y = 6$ and $4x - y + 4 = 0$ lies in

(a) I quadrant (b) II quadrant
(c) III quadrant (d) IV quadrant

3. A line passes through (2, 0). The slope of the line, for which its intercept between $y = x - 1$ and $y = -x + 1$ subtends a right angle at the origin, is/are

(a) $\sqrt{3}$ (b) $-\sqrt{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) 1

4. A straight line passes through the point of intersection of $x - 2y - 2 = 0$ and $2x - by - 6 = 0$ and the origin then the complete set of values of b for which the acute angle between this line and $y = 0$ is less than 45° are

(a) $(-\infty, 4) \cup (7, \infty)$
(b) $(-\infty, 5) \cup (7, \infty)$
(c) $(-\infty, 4) \cup (5, 7) \cup (7, \infty)$
(d) $(-\infty, 4) \cup (4, 5) \cup (7, \infty)$

5. A line has intercepts a, b on axes. When the axes are rotated through an angle α, the line makes equal intercepts on axes then $\tan\alpha =$

(a) $\frac{a+b}{a-b}$ (b) $\frac{a-b}{a+b}$ (c) $\frac{a}{b}$ (d) $\frac{b}{a}$

6. The values of m for which the system of equations $3x + my = m$ and $2x - 5y = 20$ has a solution satisfy the conditions $x > 0$, $y > 0$ are given by the set

(a) $\left\{m : m < \frac{-13}{2}\right\}$ (b) $\left\{m : m > \frac{17}{2}\right\}$

(c) $\left\{m : m < \frac{-13}{2} \text{ or } m > \frac{17}{2}\right\}$

(d) $\left\{m : m > 30 \text{ or } m < \frac{-15}{2}\right\}$

7. If $f(x + y) = f(x) f(y)\ \forall\ x, y \in R$ and $f(1) = 2$, then area enclosed by $3|x| + 2|y| \leq 8$ is (in sq.units)

(a) $f(4)$ (b) $\frac{1}{2}f(6)$ (c) $\frac{1}{3}f(6)$ (d) $\frac{1}{3}f(5)$

8. If $9x^2 + 2hxy + 4y^2 + 6x + 2fy - 3 = 0$ represents two parallel lines, then

(a) $h = 6, f = 2$ (b) $h = -6, f = 2$
(c) $h = 6, f = -2$ (d) none of these

9. If the area of the rhombus enclosed by the lines $lx \pm my \pm n = 0$ be 2 square units, then

(a) l, $2m$, n are in G.P (b) l, n, m are in G.P
(c) $lm = n$ (d) $ln = m$

10. If P is a point which moves inside an equilateral triangle of side length 'a' such that it is nearer to any angular bisector of the triangle than to any of its sides, then the area of the region in which P lies is __ sq. units.

(a) $a^2\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$ (b) $\frac{\sqrt{3}a^2}{2}\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$

(c) $\sqrt{3}a^2\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$ (d) a^2

11. The area of the triangle formed by the line $x + y = 3$ and the angular bisectors of pair of straight lines $x^2 - y^2 + 2y = 1$ is

(a) 8 sq.units (b) 6 sq.units
(c) 4 sq.units (d) none of these

12. Triangle is formed by the lines $x + y = 0$, $x - y = 0$ and $lx + my = 1$. If l and m vary subject to the condition $l^2 + m^2 = 1$, then the locus of its circumcentre is

(a) $(x^2 - y^2)^2 = x^2 + y^2$ (b) $(x^2 + y^2)^2 = x^2 - y^2$

(c) $(x^2 + y^2)^2 = 4x^2y^2$ (d) $(x^2 - y^2)^2 = (x^2 + y^2)^2$

13. A piece of cheese is located at (12, 10) in a coordinate plane. A mouse is at (4, –2) and is running up the line $y = -5x + 18$. At the point (a, b), the mouse starts getting farther from the cheese rather than closer to it. The value of $(a + b)$ is

(a) 6 (b) 10 (c) 18 (d) 14

14. If h denote the A.M., k denote the G.M. of the intercepts made on axes by the lines passing through (1, 1) then (h, k) lies on

(a) $y^2 = 2x$ (b) $y^2 = 4x$

(c) $y = 2x$ (d) $x + y = 2xy$

15. A straight rod of length $3l$ units slides with its ends A, B always on the x and y axes respectively then the locus of centroid of ΔOAB is

(a) $x^2 + y^2 = 3l^2$ (b) $x^2 + y^2 = l^2$

(c) $x^2 + y^2 = 4l^2$ (d) $x^2 + y^2 = 2l^2$

16. In a ΔABC, the coordinates of B are $(0, 0)$ $AB = 2$, $\angle ABC = \frac{\pi}{3}$ and the mid point of BC is (2,0). The centroid of triangle is

(a) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{5}{3}, \frac{1}{\sqrt{3}}\right)$

(c) $\left(\frac{4+\sqrt{3}}{3}, \frac{1}{\sqrt{3}}\right)$ (d) $\left(\frac{4-\sqrt{3}}{3}, \frac{1}{\sqrt{3}}\right)$

17. A ray travelling along the line $3x - 4y = 5$ after being reflected from a line 'l' travels along the line $5x + 12y = 13$. Then the equation of the line 'l' is

(a) $x + 8y = 0$ (b) $x - 8y = 0$

(c) $32x + 4y + 65 = 0$ (d) $32x - 4y + 65 = 0$

18. All points inside the triangle formed by $A(1, 3)$, $B(5, 6)$, $C(-1, 2)$ will satisfy

(a) $2x + 2y \le 0$ (b) $2x + y + 1 \ge 0$

(c) $2x + 3y - 12 \ge 0$ (d) $-2x + 11 \le 0$

19. A system of line is given as $y = m_i x + c_i$, where m_i can take any value out of 0, 1, –1 and when m_i is positive then c_i can be 1 or –1, when m_i equal 0, c_i can be 0 or 1 and when m_i equals – 1, c_i can take 0 or 2. Then the area enclosed by all these straight lines is

(a) $\frac{3}{\sqrt{2}}(\sqrt{2}-1)$ sq. units (b) $\frac{3}{\sqrt{2}}$ sq. units

(c) $\frac{3}{2}$ sq. units (d) $\frac{3}{4}$ sq. units

SECTION-II

Multiple Correct Answer Type

20. P is a point inside a ΔABC of area K $(K > 0)$. The lengths of perpendiculars drawn to the sides BC, CA, AB of lengths a, b, c are respectively P_1, P_2, P_3. $\frac{a}{P_1}+\frac{b}{P_2}+\frac{c}{P_3}$ is minimum when

(a) P is incentre of ΔABC

(b) P is equidistant to all the 3 sides

(c) $P_1 = P_2 = P_3 = \frac{2K}{a+b+c}$

(d) P is orthocentre of ΔABC

21. Equations $(b - c)x + (c - a)y + (a - b) = 0$ and $(b^3 - c^3)x + (c^3 - a^3)y + (a^3 - b^3) = 0$ will represent the same line if

(a) $b = c$ (b) $c = a$

(c) $a = b$ (d) $a + b + c = 0$

22. The equation of the line passing through (2, 3) and making an intercept of 2 units between the lines $y + 2x = 5$ and $y + 2x = 3$ is

(a) $5x - 4y + 2 = 0$ (b) $3x + 4y = 18$

(c) $x = 2$ (d) $y = 3$

23. The equation of the diagonal of the square formed by the pairs of lines $xy + 4x - 3y - 12 = 0$ and $xy - 3x + 4y - 12 = 0$ is

(a) $x - y = 0$ (b) $x + y + 1 = 0$

(c) $x + y = 0$ (d) $x - y + 1 = 0$

24. Under rotation of axes through θ, $x\cos\alpha + y\sin\alpha = P$ changes to $x\cos\beta + y\sin\beta = P$, then

(a) $\cos\beta = \cos(\alpha - \theta)$ (b) $\cos\alpha = \cos(\beta - \theta)$

(c) $\sin\beta = \sin(\alpha - \theta)$ (d) $\sin\alpha = \sin(\beta - \theta)$

25. Sides of a rhombus are parallel to the lines $x + y - 1 = 0$ and $7x - y - 5 = 0$. It is given that diagonals of the rhombus intersect at (1, 3) and one vertex 'A' of the rhombus lies on the line $y = 2x$. Then the coordinates of the vertex A are

(a) $\left(\frac{8}{5}, \frac{16}{5}\right)$ (b) $\left(\frac{7}{15}, \frac{14}{15}\right)$

(c) $\left(\frac{6}{5}, \frac{12}{5}\right)$ (d) $\left(\frac{4}{15}, \frac{8}{15}\right)$

26. Equations of the diagonals of a rectangle are $y + 8x - 17 = 0$ and $y - 8x + 7 = 0$. If the area of the rectangle is 8 sq. units, then the equation of the sides of the rectangle is/are

(a) $x = 1$ (b) $x + y = 1$
(c) $y = 9$ (d) $x - 2y = 3$

27. The lines $(m - 2)x + (2m - 5)y = 0$; $(m - 1)x + (m^2 - 7)y - 5 = 0$ and $x + y - 1 = 0$ are
(a) concurrent for three values of 'm'
(b) concurrent for one value of 'm'
(c) concurrent for no value of 'm'
(d) are parallel for $m = 3$

28. If $x^2 + 2hxy + y^2 = 0$ $(h \neq 1)$ represents the equations of the straight lines through the origin which make an angle α with the straight line $y + x = 0$, then

(a) $\sec 2\alpha = h$ (b) $\cos\alpha = \sqrt{\frac{1+h}{2h}}$

(c) $m_1 + m_2 = -2\sec 2\alpha$ (d) $\cot\alpha = \sqrt{\frac{h+1}{h-1}}$

SECTION-III

Comprehension Type

Paragraph for Question No. 29 to 31

In a ΔPQR, with $PQ = r$, $QR = p$, $PR = q$, the cosine values of the angles are given by

$$\cos P = \frac{q^2 + r^2 - p^2}{2qr}; \quad \cos Q = \frac{p^2 + r^2 - q^2}{2pr}$$

$$\cos R = \frac{p^2 + q^2 - r^2}{2pq} \text{ and the area of } \Delta PQR \text{ is}$$

$$\Delta = \frac{1}{2}pq\sin R = \frac{1}{2}qr\sin P = \frac{1}{2}pr\sin Q$$

Let $ABCD$ be a parallelogram whose diagonal equations are $AC \equiv x + 2y - 3 = 0$; $BD \equiv 2x + y - 3 = 0$. If $AC = 4$ units, and area of $ABCD = 8$ sq units, and $\angle BPC$ is acute where P is point of intersection of diagonals AC and BD, then

29. The length of other diagonal BD is

(a) $\frac{10}{3}$ (b) 2 (c) $\frac{20}{3}$ (d) $\frac{11\sqrt{2}}{3}$

30. The length of side AB is equal to

(a) $\frac{2\sqrt{58}}{3}$ (b) $\frac{4\sqrt{58}}{3}$ (c) $\frac{\sqrt{58}}{3}$ (d) $\sqrt{58}$

31. The length of BC is equal to

(a) $\frac{4\sqrt{10}}{3}$ (b) $\frac{2\sqrt{10}}{3}$ (c) $\frac{8\sqrt{10}}{3}$ (d) $\frac{\sqrt{10}}{3}$

SECTION-IV

Matrix-Match Type

32. Match the following Column I with Column II

Column I		Column II	
(A)	The distance between the lines $(x+7y)^2 + 4\sqrt{2}(x+7y) - 42 = 0$ is	(p)	2
(B)	If the sum of the distance of a point from two perpendicular lines in a plane is 1, then its locus is $\|x\| + \|y\| = k$, where k is equal to	(q)	7
(C)	If $6x + 6y + m = 0$ is acute angle bisector of lines $x + 2y + 4 = 0$ and $4x + 2y - 1 = 0$, then m is equal to	(r)	3
(D)	Area of the triangle formed by the lines $y^2 - 9xy + 18x^2 = 0$ and $y = 6$ is	(s)	1

33. Match the following:

Column I		Column II	
(A)	The value of k for which $4x^2 + 8xy + ky^2 = 9$ is the equation of a pair of straight lines, is	(p)	3
(B)	If the sum of the slopes of the lines given by $x^2 - 2Cxy - 7y^2 = 0$ is four times their product, then the value of C is	(q)	-3
(C)	If the gradient of one of the lines $x^2 + hxy + 2y^2 = 0$ is twice that of the other, then h =	(r)	2
(D)	If the lines $ax^2 + 2hxy + by^2 = 0$ are equally inclined to the lines $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$, then the value of λ can be	(s)	4

34. If $y = m_i x + \frac{1}{m_i}(i = 1, 2, 3)$ represent three straight lines whose slopes are the roots of the equation $2m^3 - 3m^2 - 3m + 2 = 0$, then
Match the following:

Column I		Column II	
(A)	Algebraic sum of the intercepts made by the lines on x-axis is	(p)	$\frac{(4\sqrt{2}+9\sqrt{5})}{4}$
(B)	Algebraic sum of the intercepts made by the lines on y-axis is	(q)	3/2

(C)	Sum of the distances of the lines from the origin is	(r)	$-21/4$
(D)	Sum of the lengths of the lines intercepted between the coordinate axes is	(s)	$\frac{(5\sqrt{2}+9\sqrt{5})}{10}$

SECTION-V

Integer Answer Type

35. $ABCD$ is a square of side length 1 unit. P and Q are points on AB and BC such that $\angle PDQ = 45°$. Find the perimeter of ΔPBQ.

36. Consider a ΔOAB formed by the points $O(0, 0)$, $A(2, 0)$, $B(1, \sqrt{3})$, $P(x, y)$ be any arbitrary interior point of ΔOAB moving in such a way that $d(P, OA) + d(P, AB) + d(P, OB) = \sqrt{3}$, where $d(P, OA)$, $d(P, AB)$, $d(P, OB)$ represents perpendicular distances of P from the sides OA, AB and OB respectively. If area of the region representing all possible positions of P is 'k' then $k\sqrt{3} =$

37. In a ΔABC, AB is parallel to y-axis, BC is parallel to x-axis, centroid is at $(2, 1)$. If median through C is $x - y = 1$, then the slope of median through A is

38. If the orthocentre of the triangle formed by $2x + 3y - 1 = 0$, $x + 2y - 1 = 0$, $ax + by - 1 = 0$ is at the origin then $\frac{b-a}{4} =$

39. The area of the rhombus $ABCD$ is 24. The equation of the diagonal BD is $4x + 3y + 2 = 0$ and $A = (3, 2)$. The length of the side of the rhombus is

40. In ΔABC, the equation of altitudes AM and BN are $x + 5y - 3 = 0$, $x + y - k = 0$.If the altitude CL is given by $3x - y - 1 = 0$, then $k =$

41. The coordinate axes are rotated through an angle θ about the origin in anticlockwise sense. If the equation $2x^2 + 3xy - 6x + 2y - 4 = 0$ changes to $aX^2 + 2hXY + bY^2 + 2gX + 2fY + c = 0$, then $(a + b)$ is equal to

42. Origin is shifted to $(1, 2)$ then the equation $y^2 - 8x - 4y + 12 = 0$ changes as $y^2 = 4ax$. Then $a =$

43. The centroid of the triangle formed by (a, b), (b, c), (c, a) is the origin and $a^3 + b^3 + c^3 = kabc$, then k is

SOLUTIONS

1. (b) : $x + y = 1$ meets x-axis at $A(1, 0)$ and y-axis at $B(0, 1)$.

The coordinates of P are $(1/2, 1/2)$ and PP_1 is perpendicular to OA.

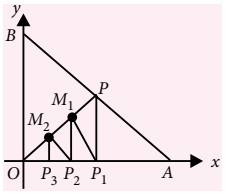
$\Rightarrow OP_1 = P_1P = 1/2$

Equation of line OP is $y = x$

We have,

$(OM_{n-1})^2 = (OP_n)^2 + (P_nM_{n-1})^2$

$= 2(OP_n)^2 = 2p_n^2$ (say)



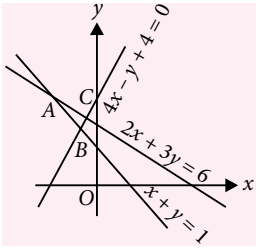
Also, $(OP_{n-1})^2 = (OM_{n-1})^2 + (P_{n-1}M_{n-1})^2$

$(OP_{n-1})^2 = (OM_{n-1})^2 + (P_{n-1}M_{n-1})^2$

$= 2p_n^2 + \frac{1}{2}p_{n-1}^2 \Rightarrow p_n^2 = \frac{1}{4}p_{n-1}^2 \Rightarrow p_n = \frac{1}{2}p_{n-1}$

$\therefore \quad Op_n = p_n = \frac{1}{2}p_{n-1} = \frac{1}{2^2}p_{n-2} = ... = \frac{1}{2^{n-1}}p_1 = \frac{1}{2^n}$

2. (a) : Coordinates of A and B are $(-3, 4)$ and $\left(-\frac{3}{5}, \frac{8}{5}\right)$.



Let orthocentre be $P(h, k)$

Then, (slope of PA) $\times$ (slope of BC) $= -1$

$\Rightarrow \frac{k-4}{h+3} \times 4 = -1$

$\Rightarrow 4k - 16 = -h - 3 \Rightarrow h + 4k = 13$ ... (i)

and slope of $PB \times$ slope of $AC = -1$

$\Rightarrow \frac{k-\frac{8}{5}}{h+\frac{3}{5}} \times \left(-\frac{2}{3}\right) = -1 \quad \Rightarrow \frac{5k-8}{5h+3} \times \frac{2}{3} = 1$

$\Rightarrow 10k - 16 = 15h + 9 \Rightarrow 15h - 10k + 25 = 0$

$\Rightarrow 3h - 2k + 5 = 0$ ... (ii)

Solving eqs. (i) and (ii), we get $h = \frac{3}{7}, k = \frac{22}{7}$

Hence, orthocentre lies in I quadrant.

3. (c) : The joint equation of straight lines $y = x - 1$ and $y = -x + 1$ is

$(x - y - 1)(x + y - 1) = 0$

$\Rightarrow x^2 - y^2 - 2x + 1 = 0$ ... (i)

Let equation of line passes through $(2, 0)$ be

$y = m(x - 2)$ ... (ii)

By homogenizing equation (i) with the help of line (ii),

we get $x^2 - y^2 - 2x\left(\frac{mx-y}{2m}\right) + \left(\frac{mx-y}{2m}\right)^2 = 0$

$\because$ coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow m = \pm\frac{1}{\sqrt{3}}$

4. (d) : As line passes through the point of intersection of $x - 2y - 2 = 0$ and $2x - by - 6 = 0$

It can be represented as

$\lambda(x - 2y - 2) + (2x - by - 6) = 0$

As it passes through the origin

$-2\lambda - 6 = 0 \Rightarrow \lambda = -3$

$\therefore$ Equation of the line is $-x + (6 - b)y = 0$

Its slope is $\frac{1}{6-b}$

As its angle with $y = 0$ is less than $\pi/4$

$\therefore -1 < \frac{1}{6-b} < 1$

$\Rightarrow 6 - b > 1$ or $b - b < -1 \Rightarrow b < 5$ or $b > 7$

But $b \neq 4$ (as the lines intersect)

$\therefore b \in (-\infty, 4) \cup (4, 5) \cup (7, \infty)$

5. (b) : Equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$

Transformed equation is

$$\frac{1}{a}(x\cos\alpha - y\sin\alpha) + \frac{1}{b}(x\sin\alpha + y\cos\alpha) = 1$$

Since, intercepts are equal

$\therefore$ x-coefficient $\equiv$ y-coefficient

$\therefore \tan\alpha = \frac{a-b}{a+b}$

6. (d) : Solving the equations, we get

$$x = \frac{25m}{2m+15},\ y = \frac{2m-60}{2m+15}$$

But $x > 0$, $y > 0$

$\Leftrightarrow 25m > 0,\ 2m + 15 > 0,\ 2m - 60 > 0$

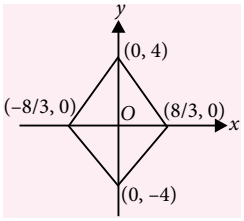
or $25\ m < 0,\ 2m + 15 < 0,\ 2m - 60 < 0$

$\Leftrightarrow m > 30$ or $m < \frac{-15}{2}$

7. (c) : Area $= 4 \times \frac{1}{2} \times \frac{8}{3} \times 4$

$= \frac{64}{3} = \frac{2^6}{3}$

$f(x) = 2^x$



8. (a) : Since the given equation represents a pair of parallel lines, we have $h^2 = ab \Rightarrow h = \pm 6$

Condition for pair of lines $\begin{vmatrix} 9 & h & 3 \\ h & 4 & f \\ 3 & f & -3 \end{vmatrix} = 0$

$\Rightarrow 108 \pm 36f - 9f^2 - 144 = 0$

$\Rightarrow f = 2$ and $h = 6$

$\Rightarrow f = -2,\ h = -6$

9. (b) : By solving the sides of the rhombus, the vertices are

$\left(0, \frac{-n}{m}\right), \left(\frac{-n}{l}, 0\right), \left(0, \frac{n}{m}\right), \left(\frac{n}{l}, 0\right)$

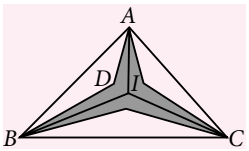
$\therefore$ Area $= \frac{1}{2}\left(\frac{2n}{m}\right)\left(\frac{2n}{l}\right) = 2 \Rightarrow n^2 = lm$

10. (b) : Shaded area is the region traced by P

$= \Delta ABC - 3\Delta ABD$

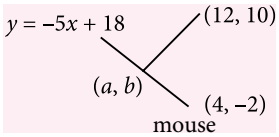
$= \frac{\sqrt{3}}{4}a^2 - \frac{3}{2}a \times \frac{a}{2}\tan 15°$

$= \frac{\sqrt{3}}{2}a^2\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$



11. (d)

12. (a)



13. (b) : $a = 2$,

$b = 8$

$a + b = 10$

14. (a) : $a = x$-intercept, $b = y$-intercept

$\therefore 2h = a + b,\ k^2 = ab$

$\frac{x}{a} + \frac{y}{b} = 1$, substitute $(1, 1)$

$\Rightarrow \frac{1}{a} + \frac{1}{b} = 1 \Rightarrow a + b = ab$

$\Rightarrow 2h = k^2 \Rightarrow y^2 = 2x$

15. (b) : Let $OA = a$, $OB = b$, $AB = 3l$

$A = (a, 0)$, $B = (0, b)$

Let $G(x, y) = \left(\frac{a}{3}, \frac{b}{3}\right)$, $a = 3x$, $b = 3y$

$a^2 + b^2 = 9l^2 \Rightarrow x^2 + y^2 = l^2$

16. (b) : Let $A(h, k)$ then $\cos 60° = \frac{h}{2} \Rightarrow h = 1$

$\sin 60° = \frac{k}{2} \Rightarrow k = \sqrt{3}$

$\therefore A(1, \sqrt{3})$ and Centroid $= \left(\frac{5}{3}, \frac{\sqrt{3}}{3}\right)$

17. (b) : The line 'l' can be any one of the bisectors of the angles between the lines $3x - 4y = 5$ and $5x + 12y = 13$

$\therefore$ Angular bisectors, $\frac{3x-4y-5}{5} = \pm\left[\frac{5x+12y-13}{13}\right]$

$\Rightarrow x - 8y = 0,\quad 32x + 4y - 65 = 0$

18. (b) : $L_1 \equiv 2x + 2y = 0$

$L_1(1, 3) > 0 \Rightarrow$ (a) is wrong.

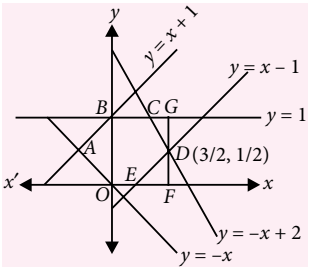
$L_2 \equiv 2x + y + 1 = 0$

$L_2(1, 3) > 0$

$L_2(5, 6) > 0 \Rightarrow$ (b) is true.

$L_2(-1, 2) > 0$

19. (c) : Lines are $y = 1$, $y = 0, y = -x, y = -x + 2$, $y = x + 1, y = x - 1$



Area of $OABCDEO$
= area of $OBGF$
$= \frac{3}{2} \times 1 = \frac{3}{2}$ sq. units

20. (a, b, c, d) : Given, $\frac{1}{2}(aP_1 + bP_2 + cP_3) = K$

$\Rightarrow \quad y = \frac{a}{P_1} + \frac{b}{P_2} + \frac{c}{P_3}$ is minimum.

When $y = \frac{1}{2K}(aP_1 + bP_2 + cP_3)\left(\frac{a}{P_1} + \frac{b}{P_2} + \frac{c}{P_3}\right)$ is minimum.

but $y = \frac{1}{2K}\left(a^2 + b^2 + c^2 + ab\left(\frac{P_1}{P_2} + \frac{P_2}{P_1}\right) + bc\left(\frac{P_2}{P_3} + \frac{P_3}{P_2}\right) + ca\left(\frac{P_1}{P_3} + \frac{P_3}{P_1}\right)\right)$

$\geq \frac{1}{2K}(a^2 + b^2 + c^2 + 2ab + 2bc + 2ac)$

$\Rightarrow \quad y \geq \frac{(a+b+c)^2}{2K}$

$\Rightarrow$ y is minimum when $\frac{P_1}{P_2} = \frac{P_2}{P_1} = \frac{P_2}{P_3} = \frac{P_3}{P_2} = \frac{P_1}{P_3} = \frac{P_3}{P_1} = 1$

i.e., when $P_1 = P_2 = P_3$

$\therefore$ P is incentre of ΔABC.

21. (a, b, c, d) : The two lines will be identical if there exists some real number k, such that

$b^3 - c^3 = k(b - c)$, $c^3 - a^3 = k(c - a)$

and $a^3 - b^3 = k(a - b)$

$\Rightarrow$ $b - c = 0$ or $b^2 + c^2 + bc = k$,

$c - a = 0$ or $c^2 + a^2 + ca = k$

and $a - b = 0$ or $a^2 + b^2 + ab = k$

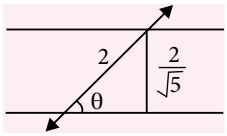
i.e., $b = c$ or $c = a$ or $a = b$

Next $b^2 + c^2 + bc = c^2 + a^2 + ca$

$\Rightarrow$ $b^2 - a^2 = c(a - b)$

Hence, $a = b$ or $a + b + c = 0$

22. (b, c) : $\tan\theta = \frac{1}{2}$



Let slope of required line be m

Slope of given lines = -2

$\left|\frac{m+2}{1-2m}\right| = \frac{1}{2} \Rightarrow m = \frac{-3}{4}$ or ∞

The lines are $3x + 4y - 18 = 0$, $x - 2 = 0$

23. (a, b) : $(x - 3)(y + 4) = 0$, $(x + 4)(y - 3) = 0$

The vertices are $A(-4, -4)$, $B(-4, 3)$, $C(3, 3)$, $D(3, -4)$

Diagonal AC is $x = y$,Diagonal BD is $x + y + 1 = 0$

24. (a, c) : $x\cos\alpha + y\sin\alpha = P$

Axis rotated through an angle 'θ'

Transformed equation is

$\cos\alpha(x\cos\theta - y\sin\theta) + \sin\alpha(x\sin\theta + y\cos\theta) = P$

or $x\cos(\alpha - \theta) + y\sin(\alpha - \theta) = P$

$\Rightarrow$ $x\cos\beta + y\sin\beta = P$

$\cos\beta = \cos(\alpha - \theta)$, $\sin\beta = \sin(\alpha - \theta)$

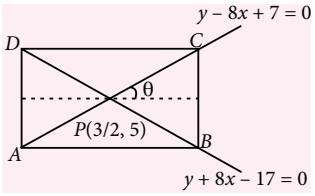
25. (a, c) : It is clear that diagonals of the rhombus will be parallel to the bisectors of the given lines and will pass through (1, 3). Equations of bisectors of the given lines are

$$\frac{x + y - 1}{\sqrt{2}} = \pm\left(\frac{7x - y - 5}{5\sqrt{2}}\right)$$

$\Rightarrow$ $2x - 6y = 0$ and $6x + 2y = 5$

Therefore, the equations of diagonals are $x - 3y + 8 = 0$ and $3x + y - 6 = 0$. Thus the required vertex will be the point where these lines meet the line $y = 2x$. Solving these lines, we get possible coordinates as $\left(\frac{8}{5}, \frac{16}{5}\right)$ and $\left(\frac{6}{5}, \frac{12}{5}\right)$.

26. (a, c) : The intersection point of the given diagonals is $P \equiv \left(\frac{3}{2}, 5\right)$



Equation of angular bisectors of the diagonals are

$$\frac{y + 8x - 17}{\sqrt{65}} = \pm\frac{y - 8x + 7}{\sqrt{65}}$$

$\Rightarrow \quad x = \frac{3}{2}$ and $y = 5$

Let length of BC be a and that of CD be b

Then $\tan\theta = \frac{a/2}{b/2} = \frac{a}{b} = 8$

Also $ab = 8 \quad \Rightarrow \quad a = 8, b = 1$

So, equations of sides are $y = 1$, $y = 9$, $x = 1$ and $x = 2$.

27. (c, d) : $\Delta = \begin{vmatrix} m-2 & 2m-5 & 0 \\ m-1 & m^2-7 & -5 \\ 1 & 1 & -1 \end{vmatrix} = 0$

$\Rightarrow$ $(m - 3)(m^2 - m + 2) = 0$

For $m = 3$, the lines become parallel.

28. (a, b, c, d) : Let lines of $x^2 + 2hxy + y^2 = 0$ be given by

$y = m_1x$ and $y = m_2x$

$m_1 + m_2 = -2h$

Slope of $y + x$ is -1

$$\tan\alpha = \left|\frac{m_1+1}{1-m_1}\right|,\ \tan\alpha = \left|\frac{m_2+1}{1-m_2}\right|$$

$$\tan\alpha = \frac{m_1+1}{1-m_1} \text{ and } \tan\alpha = -\left(\frac{m_2+1}{1-m_2}\right)$$

(for +ve signs, both gives the same value but $m_1 \neq m_2$)

$$\Rightarrow\ m_1 = \frac{\tan\alpha-1}{\tan\alpha+1}, m_2 = \frac{\tan\alpha+1}{\tan\alpha-1}$$

$m_1 + m_2 = -2\sec2\alpha$

$\Rightarrow\ h = \sec2\alpha$

$$\Rightarrow\ \cos2\alpha = \frac{1}{h} \Rightarrow 2\cos^2\alpha - 1 = \frac{1}{h}$$

$$\Rightarrow\ \cos\alpha = \left(\frac{1+h}{2h}\right)^{1/2} \Rightarrow \cot\alpha = \sqrt{\frac{h+1}{h-1}}$$

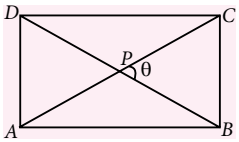
29. (c) **30. (a)** **31. (b)**

Given, P be the point of intersection

$$\tan\theta = \left|\frac{\frac{-1}{2}+2}{1+1}\right| = \frac{3}{4} \Rightarrow \sin\theta = \frac{3}{5}$$

Area of ΔCPB

$$= \frac{1}{2}PC\cdot PB\sin\theta = 2$$



$$\Rightarrow PB = \frac{10}{3} \Rightarrow BD = \frac{20}{3}$$

$$\cos(\pi-\theta) = \frac{-4}{5} = \frac{4+\frac{100}{9}-AB^2}{2\times2\times\frac{10}{3}} \Rightarrow AB = \frac{2\sqrt{58}}{3}$$

Again from ΔCPB, $BC = \frac{2\sqrt{10}}{3}$

32. A - p; B - s; C - q; D - r

(A) $(x+7y)^2 + 4\sqrt{2}(x+7y) - 42 = 0$

$$\Rightarrow\ (x+7y)[x+7y+7\sqrt{2}] - 3\sqrt{2}(x+7y) - 42 = 0$$

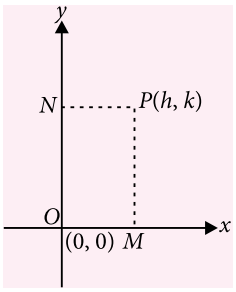
$$\Rightarrow\ (x+7y)[x+7y+7\sqrt{2}] - 3\sqrt{2}(x+7y+7\sqrt{2}) = 0$$

$$\Rightarrow\ (x+7y+7\sqrt{2})(x+7y-3\sqrt{2}) = 0$$

$x+7y+7\sqrt{2} = 0$ and $x+7y-3\sqrt{2} = 0$

$$\Rightarrow\ d = \left|\frac{7\sqrt{2}+3\sqrt{2}}{\sqrt{1+49}}\right| = \frac{10\sqrt{2}}{\sqrt{50}} = 2$$

(B) Let two perpendicular lines are coordinate axes.



Then, $PM + PN = 1$

$\Rightarrow\ h + k = 1$

Hence, the locus is $x + y = 1$

But if the point lies in other quadrants also, then

$|x| + |y| = 1$

Hence, value of k is 1.

(C) Angle bisector between the lines $x + 2y + 4 = 0$ and $4x + 2y - 1 = 0$ is

$$\frac{x+2y+4}{\sqrt{1+4}} = \pm\frac{(-4x-2y+1)}{\sqrt{16+4}}$$

$$\Rightarrow\ x+2y+4 = \pm\frac{(-4x-2y+1)}{2}$$

$$\Rightarrow\ 2(x+2y+4) = \pm(-4x-2y+1)$$

Since $aa_1 + bb_1 < 0$, so +ve sign gives acute angle bisector.

Hence, $2x + 4y + 8 = -4x - 2y + 1$

$\Rightarrow\ 6x + 6y + 7 = 0 \Rightarrow\ m = 7$

(D) We have, $y^2 - 9xy + 18x^2 = 0$

or $y^2 - 6xy - 3xy + 18x^2 = 0$

$\Rightarrow\ y(y - 6x) - 3x(y - 6x) = 0$

$\Rightarrow\ y - 3x = 0$ and $y - 6x = 0$

The third line is $y = 6$. Therefore, area of the triangle formed by these lines,

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 6 & 1 \\ 2 & 6 & 1 \end{vmatrix} = \frac{1}{2}|6-12| = 3 \text{ sq. units}$$

33. A – s; B – r; C – p, q; D – p, q, r, s

(A) The equation represents pair of lines if

$(4)(k)(-9) - (-9)(4)^2 = 0 \Rightarrow k = 4$

(B) $m_1 + m_2 = 4m_1m_2$

$$\Rightarrow\ \frac{-2h}{b} = \frac{4a}{b} \Rightarrow \frac{-2(-C)}{-7} = \frac{4\times1}{-7} \Rightarrow C = 2$$

(C) $2m_2 + m_2 = \frac{-h}{2},\ 2m_2^2 = \frac{1}{2} \Rightarrow 2\left(\frac{-h}{6}\right)^2 = \frac{1}{2}$

$\Rightarrow\ h^2 = 9 \Rightarrow\ h = \pm3$

(D) The angular bisectors of

$ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ is

$h(x^2 - y^2) - (a - b)xy = 0$

which are angular bisectors of $ax^2 + 2hxy + by^2 = 0$. The two pairs are equally inclined for any λ.

34. A - r, B - q, C - s, D - p

Solving the equation $2m^3 - 3m^2 - 3m + 2 = 0$, we get

$2(m^3 + 1) - 3m(m + 1) = 0$

$\Rightarrow (m+1)(2m^2-5m+2)=0$

$\Rightarrow (m+1)(2m-1)(m-2)=0 \Rightarrow m=-1, 1/2$ or 2

Equation of the given lines can be written as

$m_i^2x - m_iy = -1$

(A) Algebraic sum of the intercepts made by the lines on x-axis $= -\Sigma\frac{1}{m_i^2} = -\left[1+\frac{1}{4}+4\right] = -\frac{21}{4}$

(B) Algebraic sum of the intercepts made by the lines on y-axis $= \Sigma\frac{1}{m_i} = -1+2+\frac{1}{2} = \frac{3}{2}$

(C) Let p_i denote the perpendicular distance of the line from the origin, then

$$p_i = \left|\frac{1/m_i}{\sqrt{1+m_i^2}}\right| \Rightarrow \Sigma p_i = \frac{1}{\sqrt{1+1}} + \frac{2}{\sqrt{1+(1/4)}} + \frac{1/2}{\sqrt{1+4}}$$

$$= \frac{1}{\sqrt{2}} + \frac{4}{\sqrt{5}} + \frac{1}{2\sqrt{5}} = \frac{1}{10}(5\sqrt{2}+9\sqrt{5})$$

(D) l_i = length of the line intercepted between the coordinates

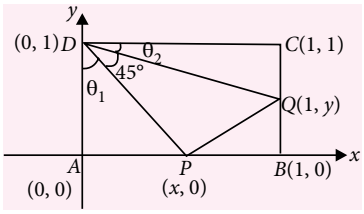
$$= \sqrt{\left(\frac{-1}{m_i^2}\right)^2 + \left(\frac{1}{m_i}\right)^2} \Rightarrow \Sigma l_i = \sqrt{1+1} + \sqrt{16+4} + \sqrt{\frac{1}{16}+\frac{1}{4}}$$

$$= \sqrt{2} + 2\sqrt{5} + \frac{\sqrt{5}}{4} = \frac{(4\sqrt{2}+9\sqrt{5})}{4}$$

35. (2) : $\tan\theta_1 = x$

and $\tan\theta_2 = 1-y$

Since, $\theta_1 + \theta_2 = 45°$



$$\Rightarrow \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2} = 1$$

$$\Rightarrow \frac{x+(1-y)}{1-x(1-y)} = 1 \Rightarrow y = \frac{2x}{1+x} \quad \ldots(i)$$

Now, perimeter $= 1-x+y+\sqrt{(1-x)^2+y^2}$

By using (i), we get Perimeter = 2

36. (3) : $ar(\Delta OAB) = ar(\Delta OPA) + ar(\Delta OPB) + ar(\Delta PAB) = \frac{\sqrt{3}}{4}\cdot 4$

Since, the triangle is an equilateral triangle,

$\therefore \quad \sqrt{3} = \frac{1}{2}2(d(P,OA)+d(P,OB)+d(P,AB))$

$\therefore$ For all positions of P

$d(P, OA) + d(P, OB) + d(P, AB) = \sqrt{3}$

$\therefore \quad k = \sqrt{3} \Rightarrow \sqrt{3}k = 3$

37. (4) : Let $B(a, b)$, $C(c, b)$, $A(a, d)$

Then D, (mid point of BC) is $\left(\frac{a+c}{2}, b\right)$

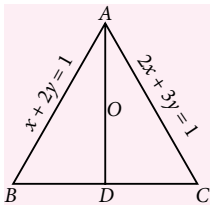
E, (mid point of AB) is $\left(a, \frac{b+d}{2}\right)$

Given slope of $CE = 1 \Rightarrow \frac{b-\frac{b+d}{2}}{c-a} = 1 \Rightarrow \frac{b-d}{c-a} = 2$

Slope of $AD = \frac{b-d}{\frac{a+c}{2}-a} = 2\frac{(b-d)}{c-a} = 4$

38. (4) : Solving $2x + 3y = 1$

$x + 2y = 1$, $A = (-1, 1)$



Orthocentre $= (0,0)$

$\Rightarrow$ slope of altitude $AD = -1$

Equation of BC is $x - y = k$

Solving, $x - y = k$

$x + 2y = 1$, we get $B = \left(\frac{1+2k}{3}, \frac{1-k}{3}\right)$

Slope of $OB = \frac{1-k}{1+2k}$, slope of $AC = -2/3$

$\therefore \quad \frac{1-k}{1+2k} = \frac{3}{2} \Rightarrow k = \frac{-1}{8}$

Equation of BC is $x - y + \frac{1}{8} = 0$

$\Rightarrow -8x + 8y - 1 = 0 \Rightarrow a = -8, b = 8$

39. (5) : Let AC and BD intersect at P

$AP = \frac{12+6+2}{\sqrt{16+9}} = 4$

Area of $\Delta ABD = AP \times BP = \frac{24}{2} = 12 \Rightarrow BP = 3$

$AB = \sqrt{AP^2+BP^2} = 5$

40. (1) : Solving the altitudes AM, CL

Orthocentre $= \left(\frac{1}{2}, \frac{1}{2}\right)$ lies on $x + y - k = 0 \Rightarrow k = 1$

41. (2) : $x = x\cos\theta - y\sin\theta$, $y = x\sin\theta + y\cos\theta$

Substitute x and y in the equation

$a = 2\cos^2\theta + 3\cos\theta\sin\theta$, $b = 2\sin^2\theta - 3\cos\theta\sin\theta$

$\Rightarrow a + b = 2$

42. (2) : Transformed equation is

$(y+2)^2 - 8(x+1) - 4(y+2) + 12 = 0$

$y^2 = 8x \Rightarrow y^2 = 4ax \Rightarrow a = 2$

43. (3) : $G = \left(\frac{a+b+c}{3}, \frac{a+b+c}{3}\right) = (0,0)$

$a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc \Rightarrow k = 3$

CONCEPT BOOSTERS

Class XII

INDEFINITE & DEFINITE INTEGRATION

*ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

INDEFINITE INTEGRATION

DEFINITION

Integration is the inverse process of differentiation.

Let $F(x)$ be a differentiable function of x such that $\frac{d}{dx}[F(x)] = f(x)$. Then $F(x)$ is called the integral of $f(x)$. It is written as $\int f(x)\,dx = F(x) + C$, where $f(x)$ is to be integrated, is called integrand and $F(x)$ is called the anti-derivative or primitive of $f(x)$, here C is known as constant of integration and can take any real value.

METHODS OF INTEGRATION

For finding the integral of complicated functions, generally these methods are used.

Integration by substitution

Integration by parts

Integration by partial fractions

Integration by Substitution:

Direct Substitution

- $\int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + C$

Let $f(x) = t \Rightarrow f'(x)dx = dt$

$\Rightarrow \int t^n dt = \frac{t^{n+1}}{n+1} + C = \frac{(f(x))^{n+1}}{n+1} + C$

- $\int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)| + C$

Let $f(x) = t \Rightarrow f'(x)dx = dt$

$\Rightarrow \int \frac{1}{t}dt = \log|t| + C = \log|f(x)| + C$

- $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + C$

Let $f(x) = t \Rightarrow dt = f'(x)dx$

$\Rightarrow \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C = 2\sqrt{f(x)} + C$

- **Standard Substitution**

In some standard integrand or a part of it, we have standard substitution. List of standard substitution is as follows:

Expression	Substitution
$x^2 + a^2$ or $\sqrt{x^2 + a^2}$	$x = a\tan\theta$ or $a\cot\theta$
$x^2 - a^2$ or $\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
$a^2 - x^2$ or $\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a+x}$ or $\sqrt{a-x}$	$x = a\cos 2\theta$
$\left(x \pm \sqrt{x^2 \pm a^2}\right)^n$	expression inside the bracket $= t$
$\sqrt{(x-a)(b-x)}$	$x = a\cos^2\theta + b\sin^2\theta$
$\frac{1}{(x+a)^{1-\frac{1}{n}}(x+b)^{1+\frac{1}{n}}}\ (n \in N, n > 1)$	$\frac{x+a}{x+b} = t$

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

Integration by Parts

If u and v be two functions of x, then integral of the product of these two functions is given by

$$\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$$

- In applying the above rule care has to be taken in the selection of the first function(u) and the second function (v). Normally we use the following methods:

1. If in the product of the two functions, one of the function is not directly integrable (*e.g.* $\ln x$, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc.) then we take it as the first function and the remaining function is taken as the second function. *e.g.* In the integration of $\int x\tan^{-1}x\,dx$, $\tan^{-1}x$ is taken as the first function and x as the second function.
2. If there is no other function, then unity is taken as the second function. *e.g.* In the integration of $\int \tan^{-1}x\,dx$, $\tan^{-1}x$ is taken as the first function and 1 as the second function.
3. If both of the functions are directly integrable, then the first function is chosen in such a way that the following preference order for the first function

(Inverse Trigonometrical, Logarithmic, Algebraic, Trigonometric, Exponential)

In the above as ILATE *e.g.* In the integration of $\int x\sin x dx$, x is taken as the first function and $\sin x$ is taken as the second function.

An important result

$$\int e^x(f(x)+f'(x))\,dx = e^x f(x)+C$$

Integration by Partial fractions

When integrand is a rational function *i.e.* of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are the polynomial functions of x, we use the method of partial fraction.

For example, we can rewrite

$$\frac{1}{(3x-1)(3x+2)} \text{ as } \frac{1}{3(3x-1)} - \frac{1}{3(3x+2)}$$

If degree of $f(x)$ is less than degree of $g(x)$ and $g(x) = (x-a_1)^{\alpha_1}.....(x^2+b_1x+c_1)^{\beta_1}.....$,

then we can put

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_1)^2} + ... + \frac{A_{\alpha_1}}{(x-a_1)^{\alpha_1}} + ...$$
$$..... + \frac{B_1x+C_1}{(x^2+b_1x+c_1)} + \frac{B_2x+C_2}{(x^2+b_1x+c_1)^2} + ...$$
$$..... + \frac{B_{\beta_1}x+C_{\beta_1}}{(x^2+b_1x+c_1)^{\beta_1}} + ...$$

Here $A_1, A_2,.....,\ A_{\alpha_1},\ .....,\ B_1, B_2,.....,B_{\beta_1}$ .....,$C_1, C_2,.....,\ C_{\beta_1}$ .....are the real constants and these can be calculated by reducing both sides of the above equation as identity in polynomial form and then by comparing the coefficients of like powers. The constants can also be obtained by putting some suitable numerical values of x in both sides of the identity.

If degree of $f(x)$ is more than or equal to degree of $g(x)$, then divide $f(x)$ by $g(x)$ so that the remainder has degree less than that of $g(x)$.

ALGEBRAIC INTEGRALS

Using the technique of standard substitution and integration by parts, we can derive the following formulae :

- $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+C$
- $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$
- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{a-x}\right|+C$
- $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+C$
- $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left[x+\sqrt{x^2+a^2}\right]+C$
- $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left[x+\sqrt{x^2-a^2}\right]+C$
- $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$
- $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})+C$
- $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln(x+\sqrt{x^2-a^2})+C$

- **Integrals of the form**

$$\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\ dx$$

Here in each case, write

$$ax^2 + bx + c = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a},$$

Put $\left(x+\frac{b}{2a}\right)=t$ and use the standard formula.

- **Integrals of the form**

$$\int \frac{(ax+b)dx}{\sqrt{cx^2+ex+f}}, \int \frac{(ax+b)dx}{cx^2+ex+f},$$

$$\int (ax+b)\sqrt{cx^2+ex+f}\ dx$$

Here write $ax + b = A(2cx + e) + B$
Find A and B by comparing the coefficients of x and constant term.

- **Integrals of the form**

$$\int \frac{(ax^2+bx+c)dx}{\sqrt{(ex^2+fx+g)}}, \int \frac{(ax^2+bx+c)dx}{(ex^2+fx+g)},$$

$$\int (ax^2+bx+c)\sqrt{(ex^2+fx+g)}\,dx$$

Here put $ax^2 + bx + c = A(ex^2 + fx + g) + B(2ex + f) + C$, find the values of A, B and C by comparing the coefficients of x^2, x and constant term.

- **Integrals of the form**

$$\int \frac{dx}{(ax+b)\sqrt{ex^2+fx+g}}$$ Here put $ax+b=\frac{1}{t}$

- **Integrals of the form**

$$\int \frac{(ax+b)dx}{(cx+d)\sqrt{ex^2+fx+g}}$$

Here put $(ax + b) = A(cx + e) + B$, find the values of A and B by comparing the coefficients of x and constant term.

- **Integrals of the form**

$$\int \frac{(ax^2+bx+c)dx}{(ex+f)\ \sqrt{gx^2+hx+i}}$$

Here put $ax^2 + bx + c = A(ex + f)\ (2gx + h) + B(ex + f) + C$, find the values of A, B and C by comparing the coefficients of x^2, x and constant term.

- **Integrals of the form**

$$\int \frac{x\,dx}{(ax^2+b)\sqrt{(cx^2+d)}}$$

Here put $cx^2 + d = t^2$.

- **Integrals of the form**

$$\int \frac{dx}{(ax^2+b)\sqrt{(cx^2+d)}}$$

Here 1st put $x=\frac{1}{t}$ and then the expression inside the square root as y^2.

- **Integrals of the form**

$$\int x^m(a+bx^n)^p\,dx\,(p\neq 0)$$

Here we have the following cases :

Case I : If p is a natural number, then expand $(a + bx^n)^p$ by binomial theorem and integrate.

Case II : If p is a negative integer and m, n are rational numbers, put $x = t^k$, when k is the LCM of denominators of m and n.

Case III : If $\frac{m+1}{n}$ is an integer and p is a rational number, put $(a + bx^n) = t^k$, when k is the denominator of p.

Case IV : If $\frac{m+1}{n}+p$ is an integer, put $\frac{a+bx^n}{x^n}=t^k$, where k is the denominator of p.

- **Integrals of the form**

$$\int \left(\frac{f(\sin x, \cos x)}{g(\sin x, \cos x)}\right) dx = \int R(\sin x, \cos x)dx,$$

where f and g both are polynomials in sin x and cos x. Here we can convert them in algebraic form by putting

$\tan\frac{x}{2}=t$ after writing

$$\sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \text{ and } \cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

- **Integrals of the form**

$$\int \left(\frac{p\sin x+q\cos x+r}{a\sin x+b\cos x+c}\right)dx$$

Here put $p \sin x + q \cos x + r = A(a \sin x + b \cos x + c) + B(a \cos x - b \sin x) + C$, values of A, B and C can be obtained by comparing the coefficients of sin$x$, cos$x$

and constant term by this technique. The given integral becomes sum of 3 integrals in which first two are very easy and in 3rd, we can put $\tan\frac{x}{2}=t$

- **Integrals of the form**

$\int(\sin^m x\cdot\cos^n x)dx$

- When $m, n\in N$

(i) If one of them is odd, then substitute for term of even power.
(ii) If both are odd, substitute either of them.
(iii) If m, n are both even, use trigonometric identities only.

- If m and n are rational numbers and $\left(\frac{m+n-2}{2}\right)$ is a negative integer, then substitute $\cot x = p$ or $\tan x = p$ which ever is found suitable.

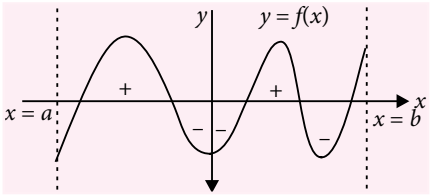
DEFINITE INTEGRATION

DEFINITION

$\int_a^b f(x)dx$ is the integration of $f(x)$ w.r.t. x with $x = a$ as lower limit and $x = b$ as upper limit.

GEOMETRICAL INTERPRETATION OF DEFINITE INTEGRAL

Let $f(x)$ be a function defined on a closed interval $[a, b]$. Then $\int_a^b f(x)dx$ represents the algebraic sum of the areas of the region bounded by the curve $y = f(x)$, x–axis and the lines $x = a$, $x = b$. The value of the definite integral may be positive, zero or negative.



FUNDAMENTAL THEOREM OF CALCULUS

If $f(x)$ is a continuous function on $[a, b]$, then

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)\ (x\in[a, b])$$

Now if we take $F'(x) = f(x)$, then by the above theorem

$$\frac{d}{dx}\left[F(x)-\int_a^x f(t)dt\right]=F'(x)-f(x)=0$$

$$\Rightarrow\quad F(x)-\int_a^x f(t)dt=\text{constant}=c\ (\text{say})$$

$$\Rightarrow\quad F(x)=\int_a^x f(t)dt+c$$

Now $F(a)=\int_a^a f(t)dt+c=0+c=c$

So, $F(x)=\int_a^x f(t)dt+F(a)$

$$\Rightarrow\quad F(b)=\int_a^b f(t)dt+F(a)$$

$$\Rightarrow\quad F(b)-F(a)=\int_a^b f(t)dt$$

Hence, if $\int f(x)dx=g(x)+c$,

then $\int_a^b f(x)dx=g(x)\Big|_a^b=g(b)-g(a)$

GENERAL PROPERTY OF DEFINITE INTEGRAL

If $\int f(x)\,dx=F(x)+C$, then $\int_a^b f(x)\,dx=F(b)-F(a)$

PERIODIC PROPERTIES OF DEFINITE INTEGRAL

- If $f(x)$ is a periodic function with period p then,

$$\int_0^{np} f(x)\,dx=n\int_0^p f(x)dx\ \forall\ n\in I$$

Proof : Let $I=\int_0^{np} f(x)dx=\int_0^p f(x)dx+\int_p^{2p} f(x)dx$
$$+\int_{2p}^{3p} f(x)dx+.....+\int_{(n-1)p}^{np} f(x)dx$$

Let $I_1=\int_{(k-1)p}^{kp} f(x)dx$

Put $x = (k - 1)p + t$
When, $x \to (k - 1)p$, $t \to 0$
and $x \to kp$, $t \to p$

$$I_1=\int_0^p f\left[(k-1)p+t\right]dt=\int_0^p f(t)dt=\int_0^p f(x)dx$$

- If $f(x)$ is a periodic function with period p, then

$$\int_{mp}^{np} f(x)dx=(n-m)\int_0^p f(x)dx\ \forall\ n, m\in I$$

DIFFERENTIATION OF DEFINITE INTEGRAL

If $F(x)=\int_{f_1(x)}^{f_2(x)} g(t)dt$,

then $F'(x) = g(f_2(x))f'_2(x) - g(f_1(x))f'_1(x)$

APPROXIMATION IN DEFINITE INTEGRAL

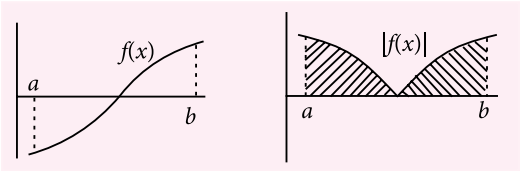
If $f_1(x) \le f(x) \le f_2(x)\ \forall\ x \in [a, b]$,

then $\int_a^b f_1(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f_2(x)dx$

If absolute maximum and minimum values of $f(x)$, when $x \in [a, b]$ is M and m respectively,

then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

$$\int_a^b f(x)dx \le \int_a^b |f(x)|dx$$



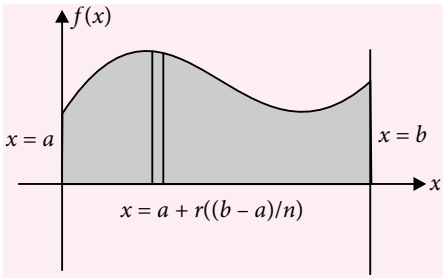
From figure it is clear.

DEFINITE INTEGRAL OF PIECEWISE CONTINUOUS FUNCTIONS

Suppose we have to evaluate $\int_a^b f(x)dx$, but either $f(x)$ is not continuous at $x = c_1, c_2, ..., c_n$ or it is not defined at these points. In both cases we have to break the limit at $c_1, c_2, ..., c_n$.

DEFINITE INTEGRAL AS THE LIMIT OF A SUM

Consider $\int_a^b f(x)dx$, for simplicity, we can take $f(x) \ge 0\ \forall\ x \in [a, b]$.



Then $\int_a^b f(x)dx$ represents the area bounded by the curve $y = f(x)$, x–axis and the lines $x = a$ and $x = b$ *i.e.* the above shaded area. Now this area can be divided into n parts.

Area of the r^{th} part can be assumed a rectangle, with width equal to $\left(\frac{b-a}{n}\right)$ and height equal to $f\left[a + r\left(\frac{b-a}{n}\right)\right]$

So that area $= \sum_{r=1}^{n}\left(\frac{b-a}{n}\right) f\left(a + r\left(\frac{b-a}{n}\right)\right)$ but this is only approximated area. To get the actual area, take rectangle with width tends to zero, hence

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{b-a}{n}\right) f\left(a + \left(\frac{b-a}{n}\right)r\right)$$

This is used in both ways *i.e.* to evaluate the definite integral as a limit of sum and also used in finding the sum of infinite terms of some series.

GAMMA FUNCTION

If n is a positive number, then the improper integral $\int_0^\infty e^{-x}x^{n-1}dx$ is defined as Gamma function and is denoted by Γn.

i.e., $\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$

Properties of Gamma Function

(i) $\Gamma 1 = 1$

(ii) $\Gamma(n+1) = n\Gamma n$

(iii) If $n \in N$, then $\Gamma(n+1) = (n)!$

(iv) $\Gamma(1/2) = \sqrt{\pi}$

Use of gamma function to find

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x\,dx$$

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x\,dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

Differentiation of function inside integral sign

If $I(\alpha) = \int_a^b f(x,\alpha)dx$, then this function can be differentiated w.r.t. α, *i.e*;

$$\frac{dI(\alpha)}{d\alpha} = \int_a^b \frac{\partial f\ (x,\alpha)}{\partial\alpha}dx$$

PROBLEMS

SECTION-I

Single Correct Answer Type

1. Let $F(x)=\sin x\int_0^x \cos t\,dt+2\int_0^x t\,dt+\cos^2 x-x^2$.

Then area bounded by $xF(x)$ and ordinate $x = 0$ and $x = 5$ with x-axis is

(a) 16 (b) $\frac{25}{2}$ (c) $\frac{35}{2}$ (d) 25

2. Area bounded by the circle which is concentric with the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ and which passes through $\left(4,-\frac{9}{5}\right)$, the vertical chord common to both circle and ellipse on the positive side of x-axis is

(a) $\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right)-\frac{36}{5}$ (b) $2\tan^{-1}\left(\frac{9}{20}\right)$

(c) $\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right)$ (d) none of these

3. Area bounded between the curves $y=\sqrt{4-x^2}$ and $y^2 = 3|x|$ is

(a) $\frac{\pi-1}{\sqrt{3}}$ (b) $\frac{2\pi-1}{3\sqrt{3}}$

(c) $\frac{2\pi-\sqrt{3}}{3}$ (d) $\frac{2\pi-\sqrt{3}}{3\sqrt{3}}$

4. The area of the smaller portion enclosed by the curves $x^2 + y^2 = 9$ and $y^2 = 8x$ is

(a) $\frac{\sqrt{2}}{3}+\frac{9\pi}{4}-\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)$

(b) $2\left(\frac{\sqrt{2}}{3}+\frac{9\pi}{4}-\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)$

(c) $2\left(\frac{\sqrt{2}}{3}+\frac{9\pi}{4}+\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)$

(d) $\frac{\sqrt{2}}{3}+\frac{9\pi}{4}+\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)$

5. If $\int\frac{\sec^2 x-2010}{\sin^{2010}x}dx=\frac{P(x)}{\sin^{2010}x}+C$, then value of $P\left(\frac{\pi}{3}\right)$ is

(a) 0 (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) 1

6. Let $I_1=\int_{\sec^2 z}^{2-\tan^2 z} x f(x(3-x))dx$

and $I_2=\int_{\sec^2 z}^{2-\tan^2 z} f(x(3-x))dx$

where 'f' is a continuous function and 'z' is any real number, then $\frac{I_1}{I_2}=$

(a) $\frac{3}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) $\frac{2}{3}$

7. $I_n=\int_0^1 x^n\tan^{-1}x\,dx$. If $a_nI_{n+2}+b_nI_n=c_n\ \forall\ n\in N$, $n \geq 1$, then

(a) $a_1, a_2, a_3, \ldots\ldots$ are in A.P.

(b) $b_1, b_2, b_3, \ldots\ldots$ are in G.P.

(c) $c_1, c_2, c_3, \ldots\ldots$ are in H.P.

(d) $a_1, a_2, a_3, \ldots\ldots$ are in H.P.

8. Let $\int_0^x\left(\frac{bt\cos 4t-a\sin 4t}{t^2}\right)dt=\frac{a\sin 4x}{x}$, then a and b are given by

(a) 1/4, 1 (b) 2, 2 (c) −1, 4 (d) 2, 4

9. $\int_{-\pi/4}^{\pi/4}\frac{e^x\sec^2 x\,dx}{e^{2x}-1}$ is equal to

(a) 0 (b) 2 (c) e (d) $2e$

10. Value of $\int_1^5\left(\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{(x-1)}}\right)dx$ is

(a) $\frac{8}{3}$ (b) $\frac{16}{3}$ (c) $\frac{32}{3}$ (d) $\frac{34}{3}$

11. If $f(x)=\int_0^x(1+t^3)^{-1/2}dt$ and $g(x)$ is the inverse of f, then the value of $\frac{g''(x)}{g^2(x)}$ is

(a) 3/2 (b) 2/3 (c) 1/3 (d) 1/2

12. $\int_{-\pi}^{\pi}\frac{2x(1+\sin x)}{1+\cos^2 x}dx=$

(a) $\frac{\pi^2}{4}$ (b) π^2 (c) 0 (d) $\frac{\pi}{2}$

13. If $\int_{\sin x}^{1} t^2 \cdot f(t)\,dt = 1-\sin x,\ \forall x \in \left(0, \frac{\pi}{2}\right)$, then the value of $f\left(\frac{1}{\sqrt{3}}\right)$ is

(a) $\frac{1}{\sqrt{3}}$ (b) $\sqrt{3}$ (c) $\frac{1}{3}$ (d) 3

14. Let $I_n = \int_0^{\infty} e^{-x}(\sin x)^n\,dx,\ n \in N, n > 1$

then $\frac{I_{2008}}{I_{2006}}$ equals

(a) $\frac{2007 \times 2006}{2008^2+1}$ (b) $\frac{2008 \times 2007}{2008^2+1}$

(c) $\frac{2006 \times 2004}{2008^2-1}$ (d) $\frac{2008 \times 2007}{2008^2-1}$

15. If $A = \int_1^{\sin\theta} \frac{t\,dt}{1+t^2}$, $B = \int_1^{\operatorname{cosec}\theta} \frac{1}{t(1+t^2)}dt$

then $\begin{vmatrix} A & A^2 & B \\ e^{A+B} & B^2 & -1 \\ 1 & A^2+B^2 & -1 \end{vmatrix} =$

(a) sin θ (b) cosec θ
(c) 0 (d) 1

16. The area of the region whose boundaries are defined by the curves $y = 2\cos x$, $y = 3\tan x$ and the y-axis, is

(a) $1+3\ln\left(\frac{2}{\sqrt{3}}\right)$ (b) $1+\frac{3}{2}\ln 3 - 3\ln 2$

(c) $1+\frac{3}{2}\ln 3 - \ln 2$ (d) $\ln 3 - \ln 2$

SECTION-II

Multiple Correct Answer Type

17. If $u = \int_0^{\infty} \frac{dx}{x^4+7x^2+1}$ and $v = \int_0^{\infty} \frac{x^2\,dx}{x^4+7x^2+1}$, then

(a) $u > v$ (b) $u < v$ (c) $u = v$ (d) $u = \frac{\pi}{6}$

18. If $f(x)$ is an odd and periodic function with period T, then

(a) $\int_0^{T} f(x)\,dx = 0$ (b) $\int_0^{3T} f(x)\,dx = 0$

(c) $\int_a^{a+T} f(x)\,dx = \int_0^{T} f(x)\,dx$ for any $a \in R$

(d) $\int_0^{T/2} f(x)\,dx = 0$

19. If the area bounded by $y = (\sin x)^{\operatorname{cosec} x}$, $y = (\operatorname{cosec} x)^{\sin x}$, $y = (\sin x)^{\sin x}$ and $y = (\operatorname{cosec} x)^{\operatorname{cosec} x}$, between the ordinates $x = 0$, $x = \frac{\pi}{2}$ denoted by A_1, A_2, A_3 and A_4 then

(a) A_4 is the greatest (b) A_1 is the least
(c) $A_2 > A_3$ (d) $A_2 < A_3$

20. Given two functions f and g which are integrable on every interval and satisfy

(i) f is odd, g is even (ii) $g(x) = f(x+5)$, then

(a) $f(x-5) = g(x)$ (b) $f(x-5) = -g(x)$

(c) $\int_0^5 f(t)dt = \int_0^5 g(5-t)dt$

(d) $\int_0^5 f(t)dt = -\int_0^5 g(5-t)dt$

21. If $\int \frac{3x+4}{x^3-2x-4}dx = \log|x-2| + k\log f(x) + c$, then

(a) $k = -1/2$ (b) $f(x) = x^2 + 2x + 2$
(c) $f(x) = |x^2 + 2x + 2|$ (d) $k = 1/4$

22. $\int_{-1/2}^{1/2} \sqrt{\left\{\left(\frac{x+1}{x-1}\right)^2 + \left(\frac{x-1}{x+1}\right)^2 - 2\right\}}dx$ is

(a) $4\ln\left(\frac{4}{3}\right)$ (b) $4\ln\left(\frac{3}{4}\right)$

(c) $-\ln\left(\frac{81}{256}\right)$ (d) $\ln\left(\frac{256}{81}\right)$

23. The value of $\int_{1/e}^{\tan x} \frac{t\,dt}{1+t^2} + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$ is

(a) $\frac{1}{2+\tan^2 x}$ (b) 1

(c) $\frac{\pi}{4}$ (d) $\frac{2}{\pi}\int_{-1}^{1} \frac{dt}{1+t^2}$

24. If $\int_0^x \left(x - [x] - \frac{1}{2}\right)dx = f(x)\cdot g(x)$, where $[x]$ and $\{x\}$ are integral and fractional parts of x, respectively then

(a) $f(x) = \frac{\{x\}}{2}$ (b) $g(x) = \{x\} - 1$

(c) $f(x) = \frac{[x]}{2}$ (d) $g(x) = [x] - 1$

25. If $I=\int \frac{5x^8+7x^6}{(x^2+1+2x^7)^2}dx$, then I is equal to

(a) $\frac{x^7}{2x^7+x^2+1}+C$ (b) $\frac{x^5}{x^2+1+2x^7}+C$

(c) $\frac{-1}{2x^7+x^2+1}+C$

(d) $\frac{p(x)}{q(x)}$, $\deg p(x)=\deg q(x)=7$

SECTION-III

Comprehension Type

Paragraph for Question No. 26 to 28

Two real valued differentiable functions $f(x)$ and $g(x)$ satisfy the following conditions:

(i) $f'(x)=\frac{f(x)-g(x)}{3}$ (ii) $g'(x)=\frac{2(g(x)-f(x))}{3}$

(iii) $f(0) = 5$ (iv) $g(0) = -1$ then

26. $\int \frac{1}{f(x)}dx=$

(a) $x - \ln (f(x)) + c$ (b) $\frac{1}{2}(x-\ln f(x))+c$

(c) $\frac{1}{3}(x-\ln f(x))+c$ (d) $\frac{1}{x-\ln(f(x))}+c$

27. $\int \frac{1}{g(x)}dx=$

(a) $x - \ln (g(x)) + c$ (b) $\frac{1}{2}(x-\ln|g(x)|)+c$

(c) $\frac{1}{3}(x-\ln|g(x)|)+c$ (d) $\frac{1}{x-\ln|g(x)|}+c$

28. $\int \frac{g(x)}{f(x)}dx=$

(a) $x + 3\ln (f(x)) + c$ (b) $x - 3\ln (f(x)) + c$

(c) $3\ln (f(x)) - x + c$ (d) $3(x - \ln (f(x)) + c$

Paragraph for Question No. 29 to 31

If $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}=A\sin^{-1}\{\sqrt{f(x)}\}+c$, then

29. The value of A is

(a) 1 (b) $\frac{1}{\sqrt{2}}$ (c) $-\frac{1}{\sqrt{2}}$ (d) 0

30. The $f(x)$ is

(a) $\frac{1+x^2}{1-x^2}$ (b) $\frac{1-x^2}{1+x^2}$ (c) $\sqrt{\frac{1-x^2}{1+x^2}}$ (d) $\sqrt{1-x^2}$

31. $\int f(x)dx$ is

(a) $2\tan^{-1}x - x + c$ (b) $2\tan^{-1}x + x + c$

(c) $\tan^{-1}x + 2x + c$ (d) none of these

Paragraph for Question No. 32 to 34

Let $f(x)$ be a differentiable function satisfying $(x - y) f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2)$ for all x, $y \in R$. If $f(1) = 1$, then

32. The function f at $x = 0$ attains

(a) Local maximum (b) Local minimum

(c) Point of inflexion (d) none of these

33. The value of $\int_{-1}^{2} f(x)dx$ is

(a) 0 (b) $\frac{1}{4}$ (c) $\frac{11}{4}$ (d) $\frac{15}{4}$

34. The area of the region bounded by the curves $y = f(x)$ and $y = x^2$ is

(a) $\frac{1}{4}$ sq. units (b) $\frac{1}{12}$ sq. units

(c) $\frac{7}{12}$ sq. units (d) $\frac{11}{12}$ sq. units

SECTION-IV

Matrix-Match Type

35. Match the following :

Column I		Column II	
(A)	$\int \sqrt{\text{cosec}\,x+1}\,dx$, $x\in\left(0,\frac{\pi}{2}\right)$	(p)	$\frac{1}{\sqrt{2}}\cos^{-1}\left(\frac{\cos x}{\sqrt{2}}\right)+c$
(B)	$\int \frac{\sin x}{\sqrt{3-\cos 2x}}dx$, $x\in\left(0,\frac{\pi}{4}\right)$	(q)	$\sin^{-1}(\sin x\cos x) + c$
(C)	$\int \frac{\cos x}{\sqrt{\cos 2x}}dx$, $x\in\left(0,\frac{\pi}{4}\right)$	(r)	$\frac{-1}{\sqrt{2}}\sin^{-1}(\sqrt{\cos 2x})+c$
(D)	$\int \frac{\cos 2x}{\sqrt{\cos^2 x+\sin^4 x}}dx$	(s)	$\sin^{-1}(2\sin x - 1) + c$

36. Match the definite integrals in column I with their values in column II.

Column I		Column II	
(A)	$\int_0^{\pi} x(\sin^2(\sin x)+\cos^2(\cos x))dx$	(p)	$\frac{\pi^2}{16}$
(B)	$\int_0^{\pi^2/4} (2\sin\sqrt{x}+\sqrt{x}\cos\sqrt{x})dx$	(q)	$\frac{\pi^2}{2}$
(C)	$\left\lvert \frac{\pi}{\ln 2}\int_{-\pi/4}^{\pi/4} \ln(\sqrt{1+\sin 2x})\,dx \right\rvert$	(r)	$\frac{\pi^2}{4}$
(D)	$\int_0^{\pi} \frac{8x^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2}dx$	(s)	$\frac{\pi^2}{8}$

SECTION-IV

Integer Answer Type

37. Let $f(x)$ be a differentiable function such that $f(x)=x^2+\int_0^x e^{-t}f(x-t)dt$, then $6f(1)=$

38. Let α, β be roots of the quadratic equation $18x^2-9\pi x+\pi^2=0$, where $\alpha<\beta$. Also $f(x)=x^2$ and $g(x)=\cos x$. If the area bounded by the curve $y=(fog)(x)$, the vertical lines $x=\alpha$, $x=\beta$ and x-axis is $\frac{\pi}{\lambda}$, then find the sum of the digits in λ.

39. Given $\int_0^{\pi/2} \ln\sin x\,dx=\frac{\pi}{2}\ln\frac{1}{2}$

and $\int_0^{\pi/2}\left(\frac{x}{\sin x}\right)^2 dx=\frac{k\cdot\pi}{2}\ln 2$, then $k=$.....

40. Find the value of $\frac{29\int_0^1(1-x^4)^7dx}{4\int_0^1(1-x^4)^6dx}$.

41. If $f(x)=a\cos(\pi x)+b$, $f'\left(\frac{1}{2}\right)=\pi$

and $\int_{1/2}^{3/2} f(x)dx=\frac{2}{\pi}+1$, then find the value of,

$-\frac{12}{\pi}(\sin^{-1}a+\cos^{-1}b)$.

42. Let $F(x)$ be a non-negative continuous function defined on R such that $F(x)+F\left(x+\frac{1}{2}\right)=3$ and the value of $\int_0^{1500} F(x)dx$ is $\frac{9000}{\lambda}$. Then the numerical value of λ is

SOLUTIONS

1. (b) : $F(x)=\sin x\int_0^x \cos t\,dt+2\int_0^x tdt-x^2+\cos^2 x$

$=\sin x(\sin t)_0^x+2\left(\frac{t^2}{2}\right)_0^x-x^2+\cos^2 x$

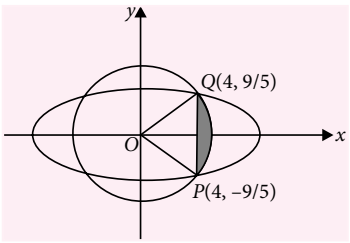
$=\sin^2 x+x^2-x^2+\cos^2 x=1$

$A=\int_0^5 xF(x)dx=\int_0^5 (x)(1)dx=\left[\frac{x^2}{2}\right]_0^5=\frac{25}{2}$

2. (a) : Eccentricity of ellipse $=\frac{4}{5}$

Co-ordinate of foci = (4, 0), (-4, 0)

$\Rightarrow \left(4,-\frac{9}{5}\right)$ is one of the end point of latus rectum



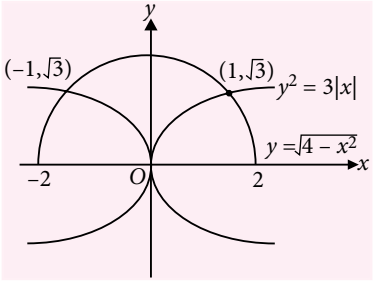
Required area is

$\frac{1}{2\pi}\times\pi\times\left(4^2+\frac{9^2}{5^2}\right)\times 2\tan^{-1}\left(\frac{9}{20}\right)-$ area of ΔPOQ

$=\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right)-\frac{1}{2}\times 4\times\left(\frac{18}{5}\right)=\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right)-\frac{36}{5}$

3. (c) : Required area $=2\int_0^1\left(\sqrt{4-x^2}-\sqrt{3x}\right)dx$

$=2\left(\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)-\frac{\sqrt{3}\cdot 2x^{3/2}}{3}\right)_0^1=\frac{2\pi-\sqrt{3}}{3}$

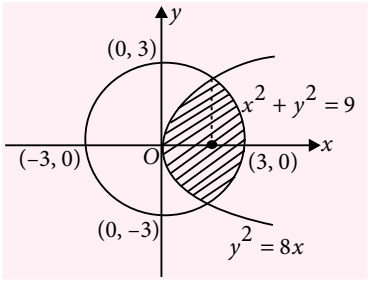


4. (b) : $x^2 + y^2 = 9$, $x^2 + 8x - 9 = 0$

$$\Rightarrow x = \frac{-8 \pm \sqrt{64+36}}{2}$$

$$\Rightarrow x = \frac{-8 \pm 10}{2} = -9, 1$$

$\Rightarrow x = 1$



$$\text{Area enclosed} = 2\left[\int_0^1 2\sqrt{2x}\,dx + \int_1^3 \sqrt{9-x^2}\,dx\right]$$

$$= 2\left[2\sqrt{2}\int_0^1 \sqrt{x}\,dx + \int_1^3 \sqrt{9-x^2}\,dx\right]$$

On simplifying, we get

$$\text{Area} = 2\left[\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

5. (c) : $\int \frac{\sec^2 x - 2010}{\sin^{2010} x}\,dx$

$$= \int \sec^2 x\,(\sin x)^{-2010} - 2010\int \frac{1}{(\sin x)^{2010}}\,dx = I_1 - I_2$$

Applying by parts on I_1, we get

$$I_1 = \frac{\tan x}{(\sin x)^{2010}} + 2010\int \frac{\tan x \cos x}{(\sin x)^{2011}}\,dx$$

$$= \frac{\tan x}{(\sin x)^{2010}} + 2010\int \frac{dx}{(\sin x)^{2010}}$$

$$\Rightarrow I = I_1 - I_2 = \frac{\tan x}{(\sin x)^{2010}} + C = \frac{P(x)}{(\sin x)^{2010}} + C$$

$$\Rightarrow P\left(\frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$$

6. (a) : $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x\,f\,(x(3-x))\,dx$

$$I_2 = \int_{\sec^2 z}^{2-\tan^2 z} f\,(x(3-x))\,dx$$

$$I_1 = \int_{\sec^2 z}^{2-\tan^2 z} (3-x)\,f((3-x)x)\,dx$$

$$\Rightarrow 2I_1 = \int_{\sec^2 z}^{2-\tan^2 z} 3f(x(3-x))\,dx$$

$$= 3\int_{\sec^2 z}^{2-\tan^2 z} f(x(3-x))\,dx \Rightarrow 2I_1 = 3I_2 \Rightarrow \frac{I_1}{I_2} = \frac{3}{2}$$

7. (a) : $I_n = \left(\frac{x^{n+1}}{n+1}\tan^{-1}x\right)_0^1 - \int_0^1 \frac{x^{n+1}}{n+1}\cdot\frac{1}{1+x^2}\,dx$

$$(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}\,dx$$

$$(n+3)I_{n+2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2}\,dx$$

$$\therefore\ (n+1)I_n + (n+3)I_{n+2} = \frac{\pi}{2} - \frac{1}{n+2}$$

$\therefore$ $a_n = (n+3) \Rightarrow a_1, a_2, a_3, \ldots\ldots$are in A.P.

$b_n = (n+1) \Rightarrow b_1, b_2, \ldots\ldots$are in A.P.

$c_n = \frac{\pi}{2} - \frac{1}{n+2}$ is not in any progression.

8. (a) : Since, $\int_0^x \left(\frac{bt\cos 4t - a\sin 4t}{t^2}\right)dt = \frac{a\sin 4x}{x}$

Differentiating both sides w.r.t. x

$$\therefore\ \frac{bx\cos 4x - a\sin 4x}{x^2} = \frac{a\{4x\cos 4x - \sin 4x\}}{x^2}$$

On comparing, we get $b = 4a$

$\therefore$ $a = 1/4$ and $b = 1$

9. (a) : Let $I = \int_{-\pi/4}^{\pi/4} \frac{e^x \sec^2 x\,dx}{e^{2x}-1}$

If $f(x) = \frac{e^x \sec^2 x}{e^{2x}-1}$

$$\therefore\ f(-x) = \frac{e^{-x}\sec^2 x}{e^{-2x}-1} = \frac{e^x \sec^2 x}{1-e^{2x}} = -\frac{e^x \sec^2 x}{e^{2x}-1} = -f(x)$$

$\therefore$ $I = 0$ ($\because$ $f(x)$ is an odd function)

10. (d) : $\because$ $\sqrt{x+(2\sqrt{(x-1)})}$

$$= \sqrt{(\sqrt{x-1})^2 + 1^2 + 2\sqrt{(x-1)}} = \sqrt{(x-1)} + 1$$

And $\sqrt{x-2\sqrt{(x-1)}} = \sqrt{(\sqrt{(x-1)})^2 + 1^2 - 2\sqrt{(x-1)}}$

$$= \left|\sqrt{(x-1)} - 1\right|$$

Then $\int_1^5 \left(\sqrt{x+2\sqrt{(x-1)}} + \sqrt{x-2\sqrt{(x-1)}}\right)dx$

$$= \int_1^5 \left(\sqrt{(x-1)}+1\right)dx + \int_1^5 \left|\sqrt{(x-1)}-1\right|dx$$

$$= \int_1^5 (\sqrt{(x-1)}+1)dx + \int_1^2 (1-\sqrt{(x-1)})dx + \int_2^5 (\sqrt{(x-1)}-1)dx$$

$$= \int_0^4 (\sqrt{x}+1)dx + \int_0^1 (1-\sqrt{x})dx + \int_1^4 (\sqrt{x}-1)dx$$

$$= \left[\frac{2}{3}\left(x^{3/2}\right)+x\right]_0^4 + \left[x-\frac{2}{3}x^{3/2}\right]_0^1 + \left[\frac{2}{3}x^{3/2}-x\right]_1^4$$

$$= \left(\frac{16}{3}+4\right)+\left(1-\frac{2}{3}\right)+\left(\frac{16}{3}-4\right)-\left(\frac{2}{3}-1\right)=\frac{34}{3}$$

11. (a) : $f(x)=\int_0^x (1+t^3)^{-1/2}dt$

i.e. $f[g(x)]=\int_0^{g(x)} (1+t^3)^{-1/2}dt$

i.e. $x=\int_0^{g(x)} (1+t^3)^{-1/2}dt$

[$\because$ g is inverse of f $\Rightarrow$ $f[g(x)] = x$]

Differentiating with respect to x, we have

$1 = (1 + g^3(x))^{-1/2}.g'(x)$

i.e. $(g'(x))^2 = 1 + g(x)^3$

Differentiating again with respect to x, we have

$$2g'(x)g''(x)=3(g(x))^2 g'(x) \Rightarrow \frac{g''(x)}{(g(x))^2}=\frac{3}{2}$$

12. (b) : $I=4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = 4\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}dx$

$$\Rightarrow 2I=4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx = 8\pi\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx$$

$\Rightarrow 2I = 2\pi^2 \Rightarrow I = \pi^2$

13. (d) : $\int_{\sin x}^{1} t^2 \cdot f(t)dt = 1-\sin x$

Differentiating both sides with respect to 'x'

$0 - \sin^2 x \cdot f(\sin x)\cdot\cos x = -\cos x$

$\Rightarrow \cos x\,[1 - \sin^2 x \cdot f(\sin x)] = 0$

But $\cos x \neq 0$

So, $f(\sin x)=\frac{1}{\sin^2 x}$ $\therefore$ $f\left(\frac{1}{\sqrt{3}}\right)=3$

14. (b) : $I_n=\int_0^{\infty} e^{-x}(\sin x)^n dx$

$$= [\sin^n x(-e^{-x})]_0^{\infty} + \int_0^{\infty} n\sin^{n-1}x\cos x\, e^{-x}dx$$

$$= 0+n\int_0^{\infty} (\sin^{n-1}x\cos x)e^{-x}dx$$

$$= n[(\sin^{n-1}x\cos x)(-e^{-x})]_0^{\infty}$$

$$-n\int_0^{\infty} \{-\sin^n x+(n-1)\sin^{n-2}x\cos^2 x\}(-e^{-x})dx$$

$$= 0+n\int_0^{\infty} e^{-x}\{-\sin^n x+(n-1)\sin^{n-2}x(1-\sin^2 x)\}dx$$

$$= n\int_0^{\infty} e^{-x}\{(n-1)\sin^{n-2}x - n\sin^n x\}dx$$

$= n(n-1)I_{n-2} - n^2 I_n \Rightarrow (1+n^2)I_n = n(n-1)I_{n-2}$

or $\frac{I_n}{I_{n-2}}=\frac{n(n-1)}{n^2+1}$

15. (c) : $B=\int_1^{\operatorname{cosec}\theta} \frac{1}{t(1+t^2)}dt$

Let $\frac{1}{t}=u \Rightarrow B=\int_1^{\sin\theta} \frac{-udu}{1+u^2} \Rightarrow A+B=0 \Rightarrow A=-B$

$$\therefore \begin{vmatrix} A & A^2 & -A \\ e^0 & A^2 & -1 \\ 1 & 2A^2 & -1 \end{vmatrix} = 0$$

16. (b) : Solving $2\cos x = 3\tan x$, we get

$2 - 2\sin^2 x = 3\sin x$

$$\Rightarrow \sin x=\frac{1}{2} \Rightarrow x=\frac{\pi}{6}$$

Required area $= \int_0^{\pi/6} (2\cos x - 3\tan x)\,dx$

$$= 2\sin x - 3\ln\sec x\,|_0^{\pi/6} = 1-3\ln 2+\frac{3}{2}\ln 3$$

17. (c, d) : $u=\int_0^{\infty} \frac{1}{x^4+7x^2+1}dx$

Put $x=\frac{1}{t}, dx=\frac{-1}{t^2}dt$

$$u=\int_{\infty}^{0}\frac{1}{\frac{1}{t^4}+\frac{7}{t^2}+1}\left(\frac{-dt}{t^2}\right)=\int_{0}^{\infty}\frac{t^2dt}{t^4+7t^2+1}=v$$

$\Rightarrow\ u=v$

So, $u+v=2u=\int_{0}^{\infty}\frac{1+x^2}{x^4+7x^2+1}dx$

$$\Rightarrow\ 2u=\int_{0}^{\infty}\frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+9}=\int_{-\infty}^{\infty}\frac{dt}{t^2+9}=\frac{\pi}{3}$$

18. (a, b, c) : Since $\int_0^T f(x)\,dx=\int_0^T f(T-x)dx$

$=\int_0^T f(-x)\,dx$ ($\because$ $f(x)$ is periodic with periodic T)

$=-\int_0^T f(x)\,dx$ ($\because$ f is odd) $\Rightarrow\ 2\int_0^T f(x)\,dx=0$

$\Rightarrow$ (a) is correct

Since $\int_0^{3T} f(x)\,dx=3\int_0^T f(x)\,dx\ \Rightarrow$ (b) is also correct.

(c) is a known result.

19. (a, b, c) : For $x\in\left(0,\frac{\pi}{2}\right)$

$0 < \sin x < 1 < \operatorname{cosec} x < \infty$

So, $\sin x^{\operatorname{cosec} x} < \sin x^{\sin x} < \operatorname{cosec} x^{\sin x} < \operatorname{cosec} x^{\operatorname{cosec} x}$

20. (b, c) : To test choice (a) and (b), we begin with computing $g(x)$. Indeed $g(x) = f(x + 5)$

$\Rightarrow\ g(-x) = f(-x + 5)$

$\Rightarrow\ g(x) = -f(x - 5)$

$\Rightarrow$ Choice (b) is true and choice (a) is ruled out. To test the choices (c) and (d), we compute

$I=\int_0^5 f(t)dt$

Indeed $I=\int_0^5 g(t-5)dt$

($\because$ $f(t) = g(t - 5)$ on replacing x by $t - 5$ in (ii))

$=\int_0^5 g(5-t)dt$ ($\because$ g is even)

$\Rightarrow$ Choice (c) is correct and choice (d) is false.

21. (a, b, c) : By using partial fraction, we get

$$\frac{3x+4}{x^3-2x-4}=\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+2}$$

Solving this equation, we get $A = 1$, $B = -1$ and $C = -1$

$$\therefore\ \int\frac{3x+4}{x^3-2x-4}dx=\int\frac{1}{x-2}dx-\int\frac{x+1}{x^2+2x+2}dx$$

$$=\log|x-2|-\frac{1}{2}\log|x^2+2x+2|+c$$

Hence $k = -1/2$, so that $f(x) = |x^2 + 2x + 2|$

22. (a, c, d) : Let $I=\int_{-1/2}^{1/2}\sqrt{\left(\frac{x+1}{x-1}-\frac{x-1}{x+1}\right)^2}dx$

$$=\int_{-1/2}^{1/2}\left|\frac{4x}{x^2-1}\right|dx=-2\int_0^{1/2}\frac{4x}{(x^2-1)}dx$$

$$=-4\left\{\ln\left|x^2-1\right|\right\}_0^{1/2}=-4\ln\left(\frac{3}{4}\right)$$

$$=4\ln\left(\frac{4}{3}\right)=\ln\left(\frac{256}{81}\right)=-\ln\left(\frac{81}{256}\right)$$

23. (b, d) : Let $I=\int_{1/e}^{\cot x}\frac{dt}{t(1+t^2)}$

Put $t=\frac{1}{z}\ \therefore\ dt=-\frac{1}{z^2}dz$

$$\therefore\ I=\int_{e}^{\tan x}\frac{-\frac{1}{z^2}dz}{\frac{1}{z}\left(1+\frac{1}{z^2}\right)}=\int_{\tan x}^{e}\frac{z\,dz}{(z^2+1)}=\int_{\tan x}^{e}\frac{t\,dt}{(1+t^2)}$$

$$\therefore\ \int_{1/e}^{\tan x}\frac{t\,dt}{(1+t^2)}+\int_{1/e}^{\cot x}\frac{dt}{t(1+t^2)}$$

$$=\int_{1/e}^{\tan x}\frac{t}{(1+t^2)}dt+\int_{\tan x}^{e}\frac{t}{(1+t^2)}dt$$

$$=\int_{1/e}^{e}\frac{t\,dt}{(1+t^2)}=\frac{1}{2}\left[\ln\left(1+t^2\right)\right]_{1/e}^{e}$$

$$=\frac{1}{2}\left\{\ln(1+e^2)-\ln\left(1+\frac{1}{e^2}\right)\right\}=\frac{1}{2}(\ln e^2)=1$$

Also, $\frac{2}{\pi}\int_{-1}^{1}\frac{dt}{1+t^2}=\frac{4}{\pi}\int_0^1\frac{dt}{1+t^2}=\frac{4}{\pi}.\tan^{-1}1=\frac{4}{\pi}.\frac{\pi}{4}=1$

24. (a, b) : $I=\int_0^{[x]+\{x\}}\left(\{x\}-\frac{1}{2}\right)dx=\int_0^{[x]}\left(\{x\}-\frac{1}{2}\right)dx+\int_{[x]}^{[x]+\{x\}}\left(\{x\}-\frac{1}{2}\right)dx$

$=[x]\int_0^1\left(\{x\}-\frac{1}{2}\right)dx+\int_0^{\{x\}}\left(\{x\}-\frac{1}{2}\right)dx$

$=[x]\int_0^1\left(x-\frac{1}{2}\right)dx+\int_0^{\{x\}}\left(x-\frac{1}{2}\right)dx$

$=[x]\left(\frac{x^2}{2}-\frac{x}{2}\right)_0^1+\left(\frac{x^2}{2}-\frac{x}{2}\right)_0^{\{x\}}$

$=0+\frac{\{x\}(\{x\}-1)}{2}=\frac{\{x\}}{2}(\{x\}-1)$

25. (a, d) : We can write

$I=\int\frac{5x^8+7x^6}{x^{14}\left(\frac{1}{x^5}+\frac{1}{x^7}+2\right)^2}dx=\int\frac{\frac{5}{x^6}+\frac{7}{x^8}}{\left(\frac{1}{x^5}+\frac{1}{x^7}+2\right)^2}dx$

Put $t=\frac{1}{x^5}+\frac{1}{x^7}+2$, so that

$I=\int\frac{-dt}{t^2}=\frac{1}{t}+C=\frac{x^7}{x^2+1+2x^7}+C$

(26 - 28) :

26. (c) : $\int\frac{1}{f(x)}dx=\int\frac{1}{2e^x+3}dx$

$=\int\frac{e^{-x}}{2+3e^{-x}}dx=\frac{-1}{3}\ln(2+3e^{-x})+c$

$=\frac{1}{3}(x-\ln f(x))+c$

27. (c) : $\int\frac{1}{g(x)}dx=\int\frac{1}{3-4e^x}dx$

$=\int\frac{e^{-x}}{3e^{-x}-4}dx=\frac{-1}{3}\ln|3e^{-x}-4|+c$

$=\frac{1}{3}(x-\ln|g(x)|)+c$

28. (b) :

$\int\frac{3-4e^x}{3+2e^x}dx=\int\left(1-\frac{3(2e^x)}{3+2e^x}\right)dx=x-3\ln(f(x))+c$

(29 - 31) :

29. (c) : $\int\frac{dx}{(1+x^2)(\sqrt{1-x^2})}=\int\frac{dx}{x^3\left(1+\frac{1}{x^2}\right)\sqrt{\frac{1}{x^2}-1}}$

Put $\frac{1}{x^2}-1=t^2\ \Rightarrow-\frac{2}{x^3}dx=2t\,dt$

$\Rightarrow-\int\frac{tdt}{(t^2+2)t}=-\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}+C=-\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{1-x^2}}{\sqrt{2}x}+C$

$=-\frac{1}{\sqrt{2}}\sin^{-1}\sqrt{\frac{1-x^2}{1+x^2}}+C\ \ \therefore\ \ A=-\frac{1}{\sqrt{2}}$

30. (b) : $f(x)=\frac{1-x^2}{1+x^2}$

31. (a) : $\int f(x)\,dx=\int\frac{1-x^2}{1+x^2}dx$

$=\int\left(\frac{2}{1+x^2}-1\right)dx=2\tan^{-1}x-x+c$

(32 - 34) :

32. (c) **33. (d)** **34. (b)**

$(x-y)f(x+y)-(x+y)f(x-y)=4xy(x^2-y^2)$

$\Rightarrow\frac{f(x+y)}{x+y}-\frac{f(x-y)}{x-y}=4xy$

So, $\frac{f(x+h)}{x+h}-\frac{f(x-h)}{x-h}=4xh$

$\Rightarrow\left[\frac{f(x+h)-f(x)+f(x)}{x+h}\right]-\left[\frac{f(x-h)-f(x)+f(x)}{x-h}\right]=4xh$

$\Rightarrow\left[\frac{f(x+h)-f(x)}{x+h}\right]-\left[\frac{f(x-h)-f(x)}{x-h}\right]=4xh+f(x)\left[\frac{1}{x-h}-\frac{1}{x+h}\right]$

$\Rightarrow\left[\frac{f(x+h)-f(x)}{x+h}\right]-\left[\frac{f(x-h)-f(x)}{x-h}\right]=4xh+\frac{2hf(x)}{x^2-h^2}$

$\Rightarrow\left[\frac{f(x+h)-f(x)}{h}\right]\cdot\left(\frac{1}{x+h}\right)+\left[\frac{f(x-h)-f(x)}{-h}\right]\left(\frac{1}{x-h}\right)=4x+\frac{2f(x)}{x^2-h^2}$

Taking limit on both sides as $h\to0$;

$\frac{f'(x)}{x}+\frac{f'(x)}{x}=4x+\frac{2f(x)}{x^2}\ \Rightarrow\ \frac{f'(x)}{x}=2x+\frac{f(x)}{x^2}$

$$f'(x)=2x^2+\frac{f(x)}{x} \Rightarrow f'(x)+\left[\frac{-f(x)}{x}\right]=2x^2,$$

which is a linear differential equation.

Integrating factor $=e^{-\int\frac{1}{x}dx}=\frac{1}{x}$

Its solution is $f(x)\cdot\frac{1}{x}=\int 2x^2\times\frac{1}{x}+c$

$\Rightarrow \frac{f(x)}{x}=x^2+c \Rightarrow f(x)=x^3+cx$

But $f(1) = 1 \Rightarrow c = 0$

So, $f(x) = x^3$

$f'(x) = 3x^2$ and $f''(x) = 6x$

Here, $f''(x) = 0 \Rightarrow x = 0$

So, at $x = 0$, $f(x)$ has point of inflexion.

$$\int_{-1}^{2}x^3dx=\int_{-1}^{1}x^3dx+\int_{1}^{2}x^3dx=0+\frac{1}{4}\left[x^4\right]_1^2=\frac{15}{4}$$

The area of the region bounded by the curves $y = f(x)$ and $y = x^2$ is

$$\int_0^1(x^2-x^3)dx=\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}\text{ sq. units}$$

35. A → s; B → p; C → r; D → q

(A) $\int\sqrt{\text{cosec}x+1}\,dx=\int\frac{\sqrt{1+\sin x}}{\sqrt{\sin x}}dx$

$=\int\frac{\cos x}{\sqrt{\sin x(1-\sin x)}}dx=\int\frac{dt}{t(1-t)}$ [Put $t=\sin x$]

$= \sin^{-1}(2\sin x - 1) + c$

(B) $\int\frac{\sin x}{\sqrt{3-\cos 2x}}dx=\frac{1}{\sqrt{2}}\int\frac{\sin x\,dx}{\sqrt{2-\cos^2 x}}$

$=\frac{1}{\sqrt{2}}\cos^{-1}\left(\frac{\cos x}{\sqrt{2}}\right)+c$

(C) $\int\frac{\cos x}{\sqrt{\cos 2x}}dx=\frac{1}{\sqrt{2}}\int\frac{\sin 2x}{\sqrt{1-\cos 2x}\sqrt{\cos 2x}}dx$

$=-\frac{1}{\sqrt{2}}\int\frac{dt}{\sqrt{1-t^2}}$ (Put $t=\sqrt{\cos 2x}$)

$=-\frac{1}{\sqrt{2}}\sin^{-1}(\sqrt{\cos 2x})+c$

(D)

$$\int\frac{\cos 2x\,dx}{\sqrt{\cos^2 x+\sin^4 x}}=\int\frac{2\cos 2x\,dx}{\sqrt{4-\sin^2 2x}}=\sin^{-1}\left(\frac{\sin 2x}{2}\right)+c$$

36. A → q; B → q; C → r; D → r

(A) $I=\int_0^{\pi}x(\sin^2(\sin x)+\cos^2(\cos x))dx$

$\Rightarrow I=\int_0^{\pi}(\pi-x)(\sin^2(\sin x)+\cos^2(\cos x))dx$

Adding $2I=\pi\int_0^{\pi}(\sin^2(\sin x)+\cos^2(\cos x))dx$

$2I=2\pi\int_0^{\pi/2}(\sin^2(\sin x)+\cos^2(\cos x))dx$

$I=\pi\int_0^{\pi/2}(\sin^2(\sin x)+\cos^2(\cos x))dx$

Also $I=\pi\int_0^{\pi/2}[\sin^2(\cos x)+\cos^2(\sin x)]dx$

Adding $2I=\pi\int_0^{\pi/2}2dx=2\pi(\pi/2)=\pi^2$

$\Rightarrow I = \pi^2/2$

(B) Let $f(x)=2\sin\sqrt{x}$

Then $x f'(x)=x.\frac{2\cos\sqrt{x}}{2\sqrt{x}}=\sqrt{x}\cos\sqrt{x}$

$I=\int_0^{\pi^2/4}(f(x)+x f'(x))dx=\left[x\ f(x)\right]_0^{\pi^2/4}$

$=2\times(\pi^2/4)\times\sin(\pi/2)=\pi^2/2$

(C) $\int_{-\pi/4}^{\pi/4}\ln\sqrt{1+\sin 2x}\ dx=\int_{-\pi/4}^{\pi/4}\ln|\sin x+\cos x|dx$

$=\int_{-\pi/4}^{\pi/4}\ln|\sqrt{2}\sin(x+\pi/4)|dx=\int_0^{\pi/2}\ln|\sqrt{2}\ \sin t|dt$

$=\ln\sqrt{2}\int_0^{\pi/2}dt+\int_0^{\pi/2}\ln\sin t\ dt$

$=(\pi/4)\ln 2-(\pi/2)\ln 2=-(\pi/4)\ \ln 2$

$\therefore \frac{\pi}{\ln 2}\int_{-\pi/4}^{\pi/4}\ln\sqrt{1+\sin 2x}\,dx=\left(\frac{\pi}{\ln 2}\right)\left(\frac{-\pi}{4}\ln 2\right)=\frac{-\pi^2}{4}$

(D) $I=\int_0^{\pi}\frac{x^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2}dx$

$I=\int_0^{\pi}\frac{(\pi-x)^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2}dx$

Adding $2I = \pi\int_0^{\pi} \cos^4 x \sin^2 x\, dx$

$I = \frac{\pi}{2}.\, 2\int_0^{\pi/2} \cos^4 x \sin^2 x\, dx = \pi\int_0^{\pi/2} \cos^4 x \sin^2 x\, dx$

$= \pi\frac{(3\cdot1)1}{6\cdot4\cdot2}(\pi/2) = \pi^2/32$

37. (8) : $f(x) = x^2 + \int_0^x e^{-t} f(x-t)dt = x^2 - e^{-x}\int_0^x e^t f(t)dt$

$\Rightarrow\ f'(x) = 2x - e^{-x}(e^x(f(x) - x^2)) + e^{-x}\cdot e^x f(x)$

$\Rightarrow\ f'(x) = 2x + x^2 \Rightarrow f(x) = \frac{x^3}{3} + x^2 + k$

But $f(0) = 0 \ \Rightarrow\ k = 0\ \ \therefore\ f(1) = \frac{4}{3}$

38. (3) : $(fog)(x) = f(\cos x) = \cos^2 x$

Let α, β be roots of the equation

$18x^2 - 9\pi x + \pi^2 = 0 \Rightarrow \alpha = \frac{\pi}{6}, \beta = \frac{\pi}{3}$

$\text{Area} = \int_{\pi/6}^{\pi/3} \cos^2 x\, dx = \frac{\pi}{12}$

39. (2) :

$I = \int_0^{\pi/2} x^2 \operatorname{cosec}^2 x\, dx = \left[-x^2\cot x\right]_0^{\pi/2} + \int_0^{\pi/2} 2x\cot x dx$

$= 0 + \lim_{x\to0^+}\frac{x^2}{\tan x} + 2\left[x\ln\sin x\right]_0^{\pi/2} - 2\int_0^{\pi/2}\ln\sin x\, dx$

$= 0 - 2\lim_{x\to0^+} x\ln\sin x - 2\frac{\pi}{2}\ln\frac{1}{2} = \pi\ln 2$

$\left(\because \lim_{x\to0^+}\frac{\ln\sin x}{1/x} = \lim_{x\to0^+}\frac{\frac{1}{\sin x}.\cos x}{-\frac{1}{x^2}} = 0\right)$

40. (7) : $I = \int_0^1 (1-x^4)^7\cdot 1\, dx$

$= \left[x\left(1-x^4\right)^7\right]_0^1 + 7\times4\int_0^1 x(1-x^4)^6 x^3 dx$

$= -28I + 28\int_0^1 (1-x^4)^6 dx \Rightarrow 29I = 28\int_0^1 (1-x^4)^6 dx$

$\Rightarrow \frac{29\int_0^1 (1-x^4)^7 dx}{4\int_0^1 (1-x^4)^6 dx} = 7$

41. (6) : $f'(x) = -a\pi\sin(\pi x)$

$\Rightarrow\ f'\left(\frac{1}{2}\right) = -a\pi\sin\frac{\pi}{2} = -a\pi = \pi \Rightarrow a = -1$

$\int_{1/2}^{3/2}(a\cos\pi x + b)dx = \left(\frac{a\sin\pi x}{\pi} + bx\right)_{1/2}^{3/2}$

$= \left(\frac{-a}{\pi} + \frac{3b}{2}\right) - \left(\frac{a}{\pi} + \frac{b}{2}\right)$

$\Rightarrow\ -\frac{2a}{\pi} + b = \frac{2}{\pi} + 1 \Rightarrow b = 1$

So, $\frac{-12}{\pi}(\sin^{-1}(-1) + \cos^{-1}1) = \frac{-12}{\pi}\left(-\frac{\pi}{2} + 0\right) = 6$

42. (4) : We have, $F(x) + F\left(x + \frac{1}{2}\right) = 3$ ....(1)

Replace x by $x + \frac{1}{2}$ in (1), we get

$F\left(x + \frac{1}{2}\right) + F(x+1) = 3$ ...(2)

From (1) and (2), we get $F(x) = F(x+1)$ .....(3)

$\Rightarrow$ $F(x)$ is periodic function.

Now consider $I = \int_0^{1500} F(x)dx = 1500\int_0^1 F(x)dx$

$= 1500\left|\int_0^{\frac{1}{2}} F(x)dx + \int_{\frac{1}{2}}^1 F(x)dx\right|$ (Using property of periodic function)

Put $x = y + \frac{1}{2}$ in 2nd integral, we get

$I = 1500\left|\int_0^{\frac{1}{2}} F(x)dx + \int_0^{\frac{1}{2}} F\left(y + \frac{1}{2}\right)dy\right|$

$= 1500\int_0^{\frac{1}{2}}\left(F(x) + F\left(x + \frac{1}{2}\right)\right)dx = 1500\int_0^{\frac{1}{2}} 3dx$ (Using (1))

Hence $I = 1500(3)\left(\frac{1}{2}\right) = 750\times3 = 2250$

WBJEE MOCK TEST PAPER

UNIT-V : VECTORS

The entire syllabus of Mathematics of WB-JEE is being divided into six units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issue.

CATEGORY I

For each correct answer one mark will be awarded, whereas, for each wrong answer, 25% of total marks (1/4) will be deducted. If candidates mark more than one answer, negative marking will be done.

1. The distance of the point $A(-2, 3, 1)$ from the line PQ through $P(-3, 5, 2)$ which makes equal angles with the axes is

(a) $\frac{2}{\sqrt{3}}$ (b) $\sqrt{\frac{14}{3}}$ (c) $\frac{16}{\sqrt{3}}$ (d) $\frac{5}{\sqrt{3}}$

2. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero vectors, no two of which are collinear. If the vector $\vec{a}+2\vec{b}$ is collinear with $\vec{c}$ and $\vec{b}+3\vec{c}$ is collinear with $\vec{a}$, then $\vec{a}+2\vec{b}+6\vec{c}$ is equal to

(a) $\lambda\vec{a}$ (b) $\lambda\vec{b}$ (c) $\lambda\vec{c}$ (d) $\vec{0}$

3. If $ABCD$ is a rhombus whose diagonals cut at the origin O, then $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}$ equals

(a) $\overrightarrow{AB}+\overrightarrow{AC}$ (b) $\vec{O}$

(c) $2(\overrightarrow{AB}+\overrightarrow{BC})$ (d) $\overrightarrow{AC}+\overrightarrow{BD}$

4. If $\vec{a}, \vec{b}$ are the position vectors of A, B respectively and C is a point on AB produced such that $AC = 3AB$, then the position vector of C is

(a) $3\vec{a}-2\vec{b}$ (b) $3\vec{b}-2\vec{a}$

(c) $3\vec{b}+2\vec{a}$ (d) $2\vec{a}-3\vec{b}$

5. If the vectors $\vec{a}+\lambda\vec{b}+3\vec{c}$, $-2\vec{a}+3\vec{b}-4\vec{c}$ and $\vec{a}-3\vec{b}+5\vec{c}$ are coplanar, then the value of λ is

(a) 2 (b) -1 (c) 1 (d) -2

6. The three points A, B, C with position vectors $-2\vec{a}+3\vec{b}+5\vec{c}$, $\vec{a}+2\vec{b}+3\vec{c}$ and $7\vec{a}-\vec{c}$ are

(a) collinear (b) non-coplanar

(c) non-collinear (d) none of these

7. If $ABCDEF$ is a regular hexagon with $\overrightarrow{AB}=\vec{a}$ and $\overrightarrow{BC}=\vec{b}$, then $\overrightarrow{CE}=$

(a) $\vec{b}-\vec{a}$ (b) $-\vec{b}$

(c) $\vec{b}-2\vec{a}$ (d) none of these

8. If $\vec{a}=\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=3\hat{i}+6\hat{j}+2\hat{k}$, then the vector in the direction of $\vec{a}$ and having magnitude $|\vec{b}|$ is

(a) $7(\hat{i}+2\hat{j}+2\hat{k})$ (b) $\frac{7}{9}(\hat{i}+2\hat{j}+2\hat{k})$

(c) $\frac{7}{3}(\hat{i}+2\hat{j}+2\hat{k})$ (d) none of these

9. Let $ABCD$ be the parallelogram whose sides AB and AD are represented by the vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\hat{i}+2\hat{j}+3\hat{k}$ respectively. If $\vec{a}$ is a unit vector parallel to $\overrightarrow{AC}$, then $\vec{a}$ is equal to

(a) $\frac{1}{3}(\hat{i}+2\hat{j}+3\hat{k})$ (b) $\frac{1}{3}(3\hat{i}+6\hat{j}+2\hat{k})$

(c) $\frac{1}{7}(3\hat{i}-6\hat{j}-3\hat{k})$ (d) $\frac{1}{7}(3\hat{i}+6\hat{j}-2\hat{k})$

10. The value of λ, for which the four points $2\hat{i}+3\hat{j}-\hat{k}$, $\hat{i}+2\hat{j}+3\hat{k}$, $3\hat{i}+4\hat{j}-2\hat{k}$ and $\hat{i}-\lambda\hat{j}+6\hat{k}$ are coplanar, is

(a) -2 (b) 8 (c) 6 (d) 0

11. Consider points A, B, C, D with position vectors $7\hat{i}-4\hat{j}+7\hat{k}$, $\hat{i}-6\hat{j}+10\hat{k}$, $-\hat{i}-3\hat{j}+4\hat{k}$ and $5\hat{i}-\hat{j}+5\hat{k}$ respectively. Then $ABCD$ is a

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(a) square (b) rhombus
(c) rectangle (d) none of these

12. The angle between the two vectors $\hat{i}+\hat{j}+\hat{k}$ and $2\hat{i}-2\hat{j}+2\hat{k}$ is equal to

(a) $\cos^{-1}\left(\frac{2}{3}\right)$ (b) $\cos^{-1}\left(\frac{1}{6}\right)$

(c) $\cos^{-1}\left(\frac{5}{6}\right)$ (d) $\cos^{-1}\left(\frac{1}{3}\right)$

13. If $\vec{p}$ and $\vec{q}$ are non-collinear unit vectors and $|\vec{p}+\vec{q}|=\sqrt{3}$, then $(2\vec{p}-3\vec{q})\cdot(3\vec{p}+\vec{q})$ is equal to

(a) 0 (b) 1/3 (c) −1/3 (d) −1/2

14. If $\vec{\alpha}=3\hat{i}-\hat{k}$, $|\vec{\beta}|=\sqrt{5}$ and $\vec{\alpha}\cdot\vec{\beta}=3$, then the area of the parallelogram for which $\vec{\alpha}$ and $\vec{\beta}$ are adjacent sides is

(a) $\frac{\sqrt{17}}{2}$ (b) $\frac{\sqrt{14}}{2}$ (c) $\frac{\sqrt{7}}{2}$ (d) $\sqrt{41}$

15. If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ then $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=$

(a) 3/2 (b) −3/2 (c) 2/3 (d) 1/2

16. If $\vec{a}\perp\vec{b}$ and $(\vec{a}+\vec{b})\perp(\vec{a}+m\vec{b})$, then $m=$

(a) −1 (b) 1

(c) $\frac{-|\vec{a}|^2}{|\vec{b}|^2}$ (d) $\frac{1}{2}$

17. If the sum of two unit vectors is also a unit vector, then the angle between the two unit vectors is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{2\pi}{3}$

18. If $\hat{i}+\hat{j}$, $\hat{j}+\hat{k}$, $\hat{i}+\hat{k}$ are the position vectors of the vertices of a triangle *ABC* taken in order, then angle *A* is equal to

(a) π/2 (b) π/5 (c) π/6 (d) π/3

19. Let $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$. If $\vec{b}$ is a vector such that $\vec{a}\cdot\vec{b}=|\vec{b}|^2$ and $|\vec{a}-\vec{b}|=\sqrt{7}$, then $|\vec{b}|=$

(a) $\sqrt{7}$ (b) $\sqrt{3}$ (c) 7 (d) 3

20. If $\vec{a},\vec{b}$ and $\vec{c}$ are three non-zero vectors such that each one of them being perpendicular to the sum of the other two vectors, then the value of $|\vec{a}+\vec{b}+\vec{c}|$ is

(a) $|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2$

(b) $|\vec{a}|+|\vec{b}|+|\vec{c}|$

(c) $2\left(|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2\right)$

(d) $\frac{1}{2}\left(|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2\right)$

21. If $\vec{\alpha}=2\hat{i}+\hat{j}+\hat{k},\vec{\beta}=\hat{i}-2\hat{j}+2\hat{k}$ and $\vec{\gamma}=3\hat{i}-4\hat{j}+2\hat{k}$, then the projection of $(\vec{\alpha}+\vec{\gamma})$ in the direction of $\vec{\beta}$ is

(a) $\frac{17}{3}$ (b) $\frac{5}{3}$ (c) $\frac{17}{\sqrt{43}}$ (d) $-\frac{1}{8}$

22. If $\vec{a},\vec{b}$ and $\vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=3,|\vec{b}|=5,|\vec{c}|=7$, the angle between $\vec{a}$ and $\vec{b}$ is

(a) 60° (b) 30° (c) 45° (d) 90°

23. If $\vec{a}=\hat{i}+2\hat{j}+2\hat{k},|\vec{b}|=5$ and angle between $\vec{a}$ and $\vec{b}$ is π/6, then the area of the triangle formed by these two vectors as two sides is

(a) $\frac{15}{4}$ (b) $\frac{15}{2}$ (c) 15 (d) $\frac{15\sqrt{3}}{2}$

24. If $\vec{a}$ and $\vec{b}$ are two unit vectors inclined at an angle π/3, then the value of $|\vec{a}+\vec{b}|$ is

(a) equal to 1 (b) greater than 1
(c) equal to 0 (d) less than 1

25. Let $\vec{a},\vec{b},\vec{c}$ be three vectors such that $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{c}\times\vec{a}=\vec{b}$, then

(a) $\vec{a}\cdot\vec{b}=|\vec{c}|^2$ (b) $\vec{c}\cdot\vec{a}=|\vec{b}|^2$
(c) $\vec{b}\cdot\vec{c}=|\vec{a}|^2$ (d) $\vec{a}\,||\,(\vec{b}\times\vec{c})$

26. Let $\vec{a}$ and $\vec{b}$ be two vectors of equal magnitude inclined at an angle θ, then $a\sin\frac{\theta}{2}$ is equal to

(a) $\frac{|\vec{a}-\vec{b}|}{2}$ (b) $\frac{|\vec{a}+\vec{b}|}{2}$

(c) $|\vec{a}-\vec{b}|$ (d) $|\vec{a}+\vec{b}|$

27. $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$, $|\vec{a}|=2,|\vec{b}|=3$, $|\vec{c}|=4$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{2\pi}{3}$, then $[\vec{a}\ \vec{b}\ \vec{c}]$ is equal to

(a) $4\sqrt{3}$ (b) $6\sqrt{3}$ (c) $12\sqrt{3}$ (d) $18\sqrt{3}$

28. The unit vector perpendicular to $\hat{i}-\hat{j}$ and coplanar with $\hat{i}+2\hat{j}$ and $2\hat{i}+3\hat{j}$ is

(a) $\frac{2\hat{i}-5\hat{j}}{\sqrt{29}}$ (b) $2\hat{i}+5\hat{j}$

(c) $\frac{\hat{i}+\hat{j}}{\sqrt{2}}$ (d) $\hat{i}+\hat{j}$

29. If $\vec{a}=\hat{i}-\hat{k}$, $\vec{b}=x\hat{i}+\hat{j}+(1-x)\hat{k}$ and $\vec{c}=y\hat{i}+x\hat{j}+(1+x-y)\hat{k}$, then $[\vec{a}\ \vec{b}\ \vec{c}]$ depends on

(a) neither x nor y (b) both x and y

(c) only x (d) only y

30. If $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=144$ and $|\vec{a}|=4$, then $|\vec{b}|$ is equal to

(a) 12 (b) 3 (c) 8 (d) 4

CATEGORY II

Every correct answer will yield 2 marks. For incorrect response, 25% of full mark (1/2) would be deducted. If candidates mark more than one answer, negative marking will be done.

31. The vector of magnitude 12, which is perpendicular to both the vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$ is

(a) $-4\hat{i}+8\hat{j}+8\hat{k}$ (b) $-2\hat{i}+4\hat{j}+4\hat{k}$

(c) $-6\hat{i}+12\hat{j}+12\hat{k}$ (d) none of these

32. $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=1$, $|\vec{b}|=4$, $\vec{a}\cdot\vec{b}=2$. If $\vec{c}=2\vec{a}\times\vec{b}-3\vec{b}$, then the angle between $\vec{b}$ and $\vec{c}$ is

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

33. Let $\overrightarrow{OA}=\vec{a}, \overrightarrow{OB}=10\vec{a}+2\vec{b}, \overrightarrow{OC}=\vec{b}$, where O, A, C are non-collinear points. Let p denotes the area of the quadrilateral $OABC$ and q denotes the area of the parallelogram with OA and OC as adjacent sides. If $p = kq$, then $k =$

(a) 2 (b) 3 (c) 6 (d) 4

34. The value of λ(constant) when $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+\lambda\hat{j}+5\hat{k}$ be coplanar, is

(a) 2 (b) –4

(c) 4 (d) none of these

35. If $[\vec{\alpha}+\vec{\beta}\ \ \vec{\beta}+\vec{\gamma}\ \ \vec{\gamma}+\vec{\alpha}]=\lambda[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}]$, then λ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

CATEGORY III

In this section more than 1 answer can be correct. Candidates will have to mark all the correct answers, for which 2 marks will be awarded. If, candidates mark one correct and one incorrect answer, then no marks will be awarded. But if, candidate makes only correct, without marking any incorrect, formula below will be used to allot marks.

2 × (no. of correct response/total no. of correct options)

36. If $\vec{a}$ is any vector, then

(a) $(\vec{a}\cdot\hat{i})\hat{i}+(\vec{a}\cdot\hat{j})\hat{j}+(\vec{a}\cdot\hat{k})\hat{k}=\vec{a}$

(b) $(\vec{a}\cdot\hat{i})^2+(\vec{a}\cdot\hat{j})^2+(\vec{a}\cdot\hat{k})^2=|\vec{a}|^2$

(c) $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=2\vec{a}$

(d) only (a) and (b)

37. The vectors $\vec{a}=\lambda\hat{i}+\hat{j}+2\hat{k}$, $\vec{b}=\hat{i}+\lambda\hat{j}-\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+\lambda\hat{k}$ are coplanar, if $\lambda =$

(a) –2 (b) $\sqrt{3}+1$

(c) $1-\sqrt{3}$ (d) 2

38. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors, then which of the following is equal to $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$?

(a) $(\vec{a}-\vec{b})\times(\vec{b}-\vec{c})$

(b) $(\vec{c}-\vec{b})\times(\vec{a}-\vec{c})$

(c) $(\vec{a}-\vec{b})\times(\vec{a}-\vec{c})$

(d) $\frac{1}{2}\{\vec{a}\times(\vec{b}-\vec{c})+\vec{b}\times(\vec{c}-\vec{a})+\vec{c}\times(\vec{a}-\vec{b})\}$

39. $[\vec{\alpha}\times\vec{a}\ \ \vec{\beta}\times\vec{b}\ \ \vec{\gamma}\times\vec{c}]$ is equal to

(a) $[\vec{\alpha}\ \ \vec{a}\ \ \vec{b}][\vec{\beta}\ \ \vec{\gamma}\ \ \vec{c}]-[\vec{\alpha}\ \ \vec{a}\ \ \vec{\beta}][\vec{b}\ \ \vec{\gamma}\ \ \vec{c}]$

(b) $[\vec{\alpha}\ \ \vec{a}\ \ \vec{\gamma}][\vec{c}\ \ \vec{\beta}\ \ \vec{b}]-[\vec{\alpha}\ \ \vec{a}\ \ \vec{c}][\vec{\gamma}\ \ \vec{\beta}\ \ \vec{b}]$

(c) $[\vec{\beta}\ \ \vec{b}\ \ \vec{\alpha}][\vec{a}\ \ \vec{\gamma}\ \ \vec{c}]-[\vec{\beta}\ \ \vec{b}\ \ \vec{a}][\vec{\alpha}\ \ \vec{\gamma}\ \ \vec{c}]$

(d) $[\vec{\alpha}\ \ \vec{\beta}\ \ \vec{\gamma}][\vec{a}\ \ \vec{b}\ \ \vec{c}]$

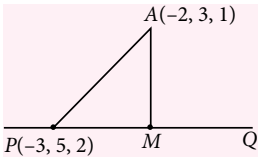
40. The vectors $\vec{a}, \vec{b}, \vec{c}$ are of same magnitude and taken pairwise, they contain equal angles. If $\vec{a}=\hat{i}+\hat{j}$, $\vec{b}=\hat{j}+\hat{k}$, then vector $\vec{c} =$

(a) $\hat{i}+\hat{k}$ (b) $\hat{j}+\hat{k}$

(c) $-\hat{i}+\hat{j}+2\hat{k}$ (d) $-\frac{1}{3}\hat{i}+\frac{4}{3}\hat{j}-\frac{1}{3}\hat{k}$

SOLUTIONS

1. (b) : Given that, the line PQ makes equal angles with the axes. Let the angle be α.



$\therefore \quad \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$\Rightarrow \quad 3\cos^2\alpha = 1$

$\Rightarrow \quad \cos\alpha = \dfrac{1}{\sqrt{3}}$

Thus, direction cosines are $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$

Now, PM = projection of AP on PQ

$$= \left|(-2+3)\frac{1}{\sqrt{3}} + (3-5)\cdot\frac{1}{\sqrt{3}} + (1-2)\cdot\frac{1}{\sqrt{3}}\right| = \frac{2}{\sqrt{3}}$$

and $AP = \sqrt{(-2+3)^2 + (3-5)^2 + (1-2)^2} = \sqrt{6}$

$$AM = \sqrt{(AP)^2 - (PM)^2} = \sqrt{6 - \frac{4}{3}} = \sqrt{\frac{14}{3}}$$

2. (d) : Given that

$\vec{a} + 2\vec{b}$ is collinear with $\vec{c}$

$\therefore \quad \vec{a} + 2\vec{b} = x\vec{c}, \ x \in R$

Also, $\vec{b} + 3\vec{c}$ is collinear with $\vec{a}$

$\therefore \quad \vec{b} + 3\vec{c} = y\vec{a}, \ y \in R$

Now, $\vec{a} + 2\vec{b} + 6\vec{c} = (x+6)\vec{c}$ ... (i)

Also, $\vec{a} + 2\vec{b} + 6\vec{c} = (1+2y)\vec{a}$ ... (ii)

From (i) and (ii), we get

$(x+6)\vec{c} = (1+2y)\vec{a}$

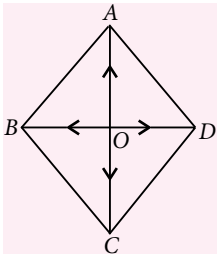
$\Rightarrow \quad (1+2y)\vec{a} - (x+6)\vec{c} = \vec{0}$

$\Rightarrow \quad 1 + 2y = 0 \quad \text{and} \quad x + 6 = 0$

$\therefore \quad y = -1/2, \quad x = -6$

$\therefore \quad \vec{a} + 2\vec{b} + 6\vec{c} = \vec{0}$

3. (b) : We know that diagonals of a rhombus bisect each other.



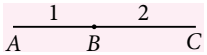
$\therefore \quad \overrightarrow{OA} = -\overrightarrow{OC}$ and $\overrightarrow{OB} = -\overrightarrow{OD}$

So, $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = \vec{O}$

4. (b) : Given that $AC = 3AB$

$\Rightarrow$ B divides AC internally in the ratio 1 : 2

Let $\vec{c}$ be the position vector of C



$\therefore \quad \vec{b} = \dfrac{1\cdot\vec{c} + 2\cdot\vec{a}}{1+2} \quad \Rightarrow \quad \vec{c} = 3\vec{b} - 2\vec{a}$

5. (d) : Given that $\vec{a} + \lambda\vec{b} + 3\vec{c}$, $-2\vec{a} + 3\vec{b} - 4\vec{c}$ and $\vec{a} - 3\vec{b} + 5\vec{c}$ are coplanar.

$\therefore$ It is possible to express one of them as a linear combinaton of the other two.

Let us write

$$\vec{a} + \lambda\vec{b} + 3\vec{c} = x(-2\vec{a} + 3\vec{b} - 4\vec{c}) + y(\vec{a} - 3\vec{b} + 5\vec{c})$$

where x, y are scalars.

Now, comparing the coefficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$ from both the sides, we get

$-2x + y = 1$, $-4x + 5y = 3$ and $3x - 3y = \lambda$

Solving first two equations, we get $x = -1/3$ and $y = 1/3$. These values of x and y satisfy the last equation as well.

$\therefore \quad 3\left(-\dfrac{1}{3}\right) - 3\left(\dfrac{1}{3}\right) = \lambda \quad \Rightarrow \quad \lambda = -2$

6. (a) : Let O be the origin of reference.

Then, $\overrightarrow{OA} = -2\vec{a} + 3\vec{b} + 5\vec{c}$

$\overrightarrow{OB} = \vec{a} + 2\vec{b} + 3\vec{c}$

$\overrightarrow{OC} = 7\vec{a} - \vec{c}$

$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (\vec{a} + 2\vec{b} + 3\vec{c}) - (-2\vec{a} + 3\vec{b} + 5\vec{c})$

$= 3\vec{a} - \vec{b} - 2\vec{c}$

Also, $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (7\vec{a} - \vec{c}) - (-2\vec{a} + 3\vec{b} + 5\vec{c})$

$= 9\vec{a} - 3\vec{b} - 6\vec{c}$

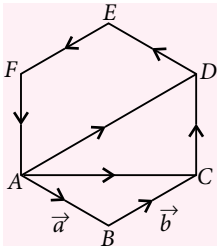
$\overrightarrow{AC} = 3(3\vec{a} - \vec{b} - 2\vec{c}) \quad \Rightarrow \quad \overrightarrow{AC} = 3\overrightarrow{AB}$

The above equation shows that the vectors $\overrightarrow{AC}$ and $\overrightarrow{AB}$ have the same or parallel supports.

But these vectors have a common initial point A. So, $\overrightarrow{AC}$ and $\overrightarrow{AB}$ have the same support.

$\therefore$ A, B, C are collinear.

7. (c): The sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of a regular hexagon $ABCDEF$ represents the vectors $\vec{a}$ and $\vec{b}$ respectively.



Then $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{b}$

Now, $\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

$\Rightarrow \quad \overrightarrow{CD} = \overrightarrow{AD} - \overrightarrow{AC}$

$= 2\vec{b} - (\vec{a} + \vec{b})$

$(\because \ \overrightarrow{AD} \parallel \overrightarrow{BC})$

$\Rightarrow \quad \overrightarrow{CD} = \vec{b} - \vec{a}$

Now in ΔCDE, $\overrightarrow{CE} = \overrightarrow{CD} + \overrightarrow{DE} = \vec{b} - \vec{a} + (-\vec{a}) = \vec{b} - 2\vec{a}$

8. (c): Here $|\vec{b}| = \sqrt{3^2 + 6^2 + 2^2} = 7$

Now, any vector along $\vec{a}$ with magnitude $|\vec{b}| = 7$ is

$$7\cdot\frac{\vec{a}}{|\vec{a}|} = \frac{7}{\sqrt{1^2 + 2^2 + 2^2}}(\hat{i} + 2\hat{j} + 2\hat{k}) = \frac{7}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$$

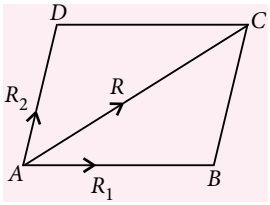
9. (d) : Let $\vec{R}_1 = 2\hat{i} + 4\hat{j} - 5\hat{k}$

and $\vec{R}_2 = \hat{i} + 2\hat{j} + 3\hat{k}$

$\therefore \quad \vec{R} = \vec{R}_1 + \vec{R}_2$

$= 2\hat{i} + 4\hat{j} - 5\hat{k} + \hat{i} + 2\hat{j} + 3\hat{k}$

$= 3\hat{i} + 6\hat{j} - 2\hat{k}$



$\therefore$ Unit vector along $\overrightarrow{AC} = \dfrac{\vec{R}}{|\vec{R}|} = \dfrac{3\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{9+36+4}}$

$= \dfrac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k})$

10. (a) : Let the four points are

$\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{B} = \hat{i} + 2\hat{j} + 3\hat{k}$,

$\vec{C} = 3\hat{i} + 4\hat{j} - 2\hat{k}$ and $\vec{D} = \hat{i} - \lambda\hat{j} + 6\hat{k}$

Now, $\overrightarrow{AB} = (\hat{i} + 2\hat{j} + 3\hat{k}) - (2\hat{i} + 3\hat{j} - \hat{k})$

$\therefore \quad \overrightarrow{AB} = -\hat{i} - \hat{j} + 4\hat{k}$,

$\overrightarrow{AC} = (3\hat{i} + 4\hat{j} - 2\hat{k}) - (2\hat{i} + 3\hat{j} - \hat{k})$

$\therefore \quad \overrightarrow{AC} = \hat{i} + \hat{j} - \hat{k}$ and $\overrightarrow{AD} = (\hat{i} - \lambda\hat{j} + 6\hat{k}) - (2\hat{i} + 3\hat{j} - \hat{k})$

$\therefore \quad \overrightarrow{AD} = -\hat{i} - (\lambda + 3)\hat{j} + 7\hat{k}$

Now, $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ will be coplanar, if

$\overrightarrow{AD} = x \cdot \overrightarrow{AB} + y \cdot \overrightarrow{AC}$ [x, y are scalars]

$\Rightarrow \quad -\hat{i} - (\lambda + 3)\hat{j} + 7\hat{k} = x(-\hat{i} - \hat{j} + 4\hat{k}) + y(\hat{i} + \hat{j} - \hat{k})$

Comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$-x + y = -1$, $4x - y = 7$ and $-x + y = -(\lambda + 3)$

Solving the first two equations, we get $x = 2$ and $y = 1$.

These values of x and y satisfy the last equation as well.

$\therefore \; -2 + 1 = -\lambda - 3 \;\Rightarrow\; \lambda = -2$

11. (d) : Given that,

$\overrightarrow{OA} = 7\hat{i} - 4\hat{j} + 7\hat{k}$, $\overrightarrow{OB} = \hat{i} - 6\hat{j} + 10\hat{k}$

$\overrightarrow{OC} = -\hat{i} - 3\hat{j} + 4\hat{k}$ and $\overrightarrow{OD} = 5\hat{i} - \hat{j} + 5\hat{k}$

Now, $|\overrightarrow{AB}| = \sqrt{(7-1)^2 + (-4+6)^2 + (7-10)^2}$

$= \sqrt{36+4+9} = 7$

$|\overrightarrow{BC}| = \sqrt{(1+1)^2 + (-6+3)^2 + (10-4)^2}$

$= \sqrt{4+9+36} = 7$

$|\overrightarrow{CD}| = \sqrt{(-1-5)^2 + (-3+1)^2 + (4-5)^2}$

$= \sqrt{36+4+1} = \sqrt{41}$

$|\overrightarrow{DA}| = \sqrt{(5-7)^2 + (-1+4)^2 + (5-7)^2}$

$= \sqrt{4+9+4} = \sqrt{17}$

which does not satisfy the conditions of square, rhombus and rectangle.

12. (d) : Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - 2\hat{j} + 2\hat{k}$

If θ is the angle between the two vectors $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{2-2+2}{\sqrt{3}\sqrt{12}} = \frac{2}{\sqrt{3} \cdot 2\sqrt{3}} = \frac{1}{3}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

13. (d) : Given that, $|\vec{p} + \vec{q}| = \sqrt{3}$

$\Rightarrow \quad |\vec{p}|^2 + |\vec{q}|^2 + 2(\vec{p} \cdot \vec{q}) = 3$

Since $\vec{p}$ and $\vec{q}$ are unit vectors

$\therefore \quad 1 + 1 + 2(\vec{p} \cdot \vec{q}) = 3 \;\Rightarrow\; \vec{p} \cdot \vec{q} = \dfrac{1}{2}$

Now, $(2\vec{p} - 3\vec{q}) \cdot (3\vec{p} + \vec{q}) = 6|\vec{p}|^2 + 2(\vec{p} \cdot \vec{q}) - 9(\vec{q} \cdot \vec{p}) - 3|\vec{q}|^2$

$= 6 - 7\left(\dfrac{1}{2}\right) - 3 = 3 - \dfrac{7}{2} = -\dfrac{1}{2}$

14. (d) : Given that $\vec{\alpha} = 3\hat{i} - \hat{k}$, $|\vec{\beta}| = \sqrt{5}$ and $\vec{\alpha} \cdot \vec{\beta} = 3$

$\therefore \quad \vec{\alpha} \cdot \vec{\beta} = |\vec{\alpha}||\vec{\beta}|\cos\theta = 3$

$\Rightarrow \quad \sqrt{10}\sqrt{5}\cos\theta = 3 \quad [\because |\vec{\alpha}| = \sqrt{3^2 + 1^2} = \sqrt{10}]$

$\Rightarrow \quad \cos\theta = \dfrac{3}{\sqrt{50}}$

Now, area of parallelogram $= |\vec{\alpha} \times \vec{\beta}|$

$= |\vec{\alpha}||\vec{\beta}|\sin\theta = \sqrt{50} \times \sqrt{1 - \dfrac{9}{50}} = \sqrt{41}$

15. (b) : Given that $\vec{a} + \vec{b} + \vec{c} = \vec{0} \;\Rightarrow\; |\vec{a} + \vec{b} + \vec{c}| = 0$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$

$\Rightarrow \quad \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -\dfrac{3}{2}$ ($\because$ $\vec{a}, \vec{b}, \vec{c}$ are unit vectors)

16. (c): Since $\vec{a} \perp \vec{b}$, $\therefore \quad \vec{a} \cdot \vec{b} = 0$

Now, $(\vec{a} + \vec{b}) \cdot (\vec{a} + m\vec{b}) = |\vec{a}|^2 + m(\vec{a} \cdot \vec{b}) + \vec{b} \cdot \vec{a} + m|\vec{b}|^2 = 0$

($\because \; (\vec{a} + \vec{b}) \perp (\vec{a} + m\vec{b})$)

$\Rightarrow |\vec{a}|^2 + (m+1)\vec{a} \cdot \vec{b} + m|\vec{b}|^2 = 0$

$\Rightarrow \quad |\vec{a}|^2 + m|\vec{b}|^2 = 0 \Rightarrow \quad m = -\dfrac{|\vec{a}|^2}{|\vec{b}|^2}$

17. (d) : Let $\hat{a}$ and $\hat{b}$ be two unit vectors.

We have, $|\hat{a}+\hat{b}|^2=1$

$\Rightarrow |\hat{a}|^2+2(\hat{a}\cdot\hat{b})+|\hat{b}|^2=1 \Rightarrow 2\hat{a}\cdot\hat{b}=-1$

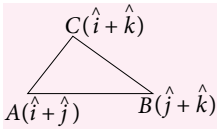
$\Rightarrow \hat{a}\cdot\hat{b}=-\frac{1}{2} \Rightarrow |\hat{a}||\hat{b}|\cos\theta=-\frac{1}{2}$

$\Rightarrow \cos\theta=\cos\left(\pi-\frac{\pi}{3}\right) \Rightarrow \theta=\frac{2\pi}{3}$

18. (d) : $\overrightarrow{AC}=(\hat{i}+\hat{k})-(\hat{i}+\hat{j})=\hat{k}-\hat{j}$

$\overrightarrow{AB}=(\hat{j}+\hat{k})-(\hat{i}+\hat{j})=\hat{k}-\hat{i}$

$\therefore \cos\theta=\frac{1-0-0}{\sqrt{2}\sqrt{2}}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}$



19. (a) : Here, $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$

$\therefore |\vec{a}|=\sqrt{1+4+9}=\sqrt{14}$

Also, $\vec{a}\cdot\vec{b}=|\vec{b}|^2$ and $|\vec{a}-\vec{b}|=\sqrt{7}$

$\therefore |\vec{a}-\vec{b}|^2=7$

$\Rightarrow |\vec{a}|^2-2(\vec{a}\cdot\vec{b})+|\vec{b}|^2=7$

$\Rightarrow 14-2|\vec{b}|^2+|\vec{b}|^2=7 \Rightarrow |\vec{b}|=\sqrt{7}$

20. (a) : $|\vec{a}+\vec{b}+\vec{c}|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2$

$+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$ ... (i)

$\vec{a}\perp(\vec{b}+\vec{c}) \Rightarrow \vec{a}\cdot(\vec{b}+\vec{c})=0$ ... (ii)

$\vec{b}\perp(\vec{c}+\vec{a}) \Rightarrow \vec{b}\cdot(\vec{c}+\vec{a})=0$ ... (iii)

and $\vec{c}\perp(\vec{a}+\vec{b}) \Rightarrow \vec{c}\cdot(\vec{a}+\vec{b})=0$ ... (iv)

Adding (ii), (iii) and (iv), we get

$2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\therefore$ (i) becomes

$|\vec{a}+\vec{b}+\vec{c}|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2$

21. (a) : Here $\vec{\alpha}=2\hat{i}+\hat{j}+\hat{k}$, $\vec{\beta}=\hat{i}-2\hat{j}+2\hat{k}$

and $\vec{\gamma}=3\hat{i}-4\hat{j}+2\hat{k}$

$\therefore \vec{\alpha}+\vec{\gamma}=5\hat{i}-3\hat{j}+3\hat{k}$

Now, projection of $(\vec{\alpha}+\vec{\gamma})$ on $\vec{\beta}$ is

$$\frac{(\vec{\alpha}+\vec{\gamma})\cdot\vec{\beta}}{|\vec{\beta}|}=\frac{5\cdot1+(-2)(-3)+2\cdot3}{\sqrt{1^2+2^2+2^2}}=\frac{17}{3}$$

22. (a) : Given that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow \vec{a}+\vec{b}=-\vec{c} \Rightarrow |\vec{a}+\vec{b}|^2=|-\vec{c}|^2$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2(\vec{a}\cdot\vec{b})=|\vec{c}|^2$

$\Rightarrow 9+25+2\cdot3\cdot5\cos\theta=49$

$\Rightarrow 30\cos\theta=15 \Rightarrow \cos\theta=\frac{1}{2}$

$\therefore \theta=60°$

23. (a) : Here $\vec{a}=\hat{i}+2\hat{j}+2\hat{k}$

$\Rightarrow |\vec{a}|=\sqrt{1^2+2^2+2^2}=3$

$\therefore$ Required area $=\frac{1}{2}|\vec{a}\times\vec{b}|=\frac{1}{2}|\vec{a}||\vec{b}|\sin\theta$

$$=\frac{1}{2}\times3\times5\sin\frac{\pi}{6}=\frac{15}{4}$$

24. (b) : Since $\vec{a}$ and $\vec{b}$ are two unit vectors,

$\therefore |\vec{a}|=|\vec{b}|=1$

Now, $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta=1\cdot1\cdot\cos\frac{\pi}{3}=\frac{1}{2}$

$|\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2+2(\vec{a}\cdot\vec{b})=3$

$\Rightarrow |\vec{a}+\vec{b}|=\sqrt{3}>1$

25. (d) : We have, $\vec{a}\times\vec{b}=\vec{c}$

$\Rightarrow \vec{a}\perp\vec{c}$ and $\vec{b}\perp\vec{c}$ ... (i)

Also, $\vec{c}\times\vec{a}=\vec{b}$

$\Rightarrow \vec{c}\perp\vec{b}$ and $\vec{a}\perp\vec{b}$ ... (ii)

From (i) and (ii), we get

$\vec{a}\perp\vec{c}$ and $\vec{a}\perp\vec{b} \Rightarrow \vec{a}\,||\,(\vec{b}\times\vec{c})$

26. (a) : Here, $|\vec{a}|=|\vec{b}|$

$\therefore |\vec{a}-\vec{b}|=\sqrt{|\vec{a}|^2+|\vec{b}|^2-2|\vec{a}||\vec{b}|\cos\theta}$

$=\sqrt{|\vec{a}|^2+|\vec{a}|^2-2|\vec{a}|^2\cos\theta}$

$=\sqrt{2a^2(1-\cos\theta)}=\sqrt{4a^2\sin^2\frac{\theta}{2}}$

$\Rightarrow |\vec{a}-\vec{b}|=2a\sin\frac{\theta}{2} \Rightarrow a\sin\frac{\theta}{2}=\frac{1}{2}|\vec{a}-\vec{b}|$

27. (c): Given that $|\vec{a}|=2, |\vec{b}|=3, |\vec{c}|=4$

$\therefore [\vec{a}\ \vec{b}\ \vec{c}]=\vec{a}\cdot(\vec{b}\times\vec{c})=\vec{a}\cdot\left(|\vec{b}||\vec{c}|\sin\frac{2\pi}{3}\hat{n}\right)$

$= |\vec{a}||\vec{b}||\vec{c}|\sin\frac{2\pi}{3}$ $[\because \vec{a}\cdot\hat{n} = |\vec{a}||\hat{n}|\cos 0° = |\vec{a}|]$

$= 2\times 3\times 4\times \frac{\sqrt{3}}{2} = 12\sqrt{3}$

28. (c): Given two vectors lie in xy-plane. Therefore, a vector coplanar with them is

$\vec{a} = a_1\hat{i} + a_2\hat{j}$

Since $\vec{a} = (\hat{i} + 2\hat{j}) + \lambda(2\hat{i} + 3\hat{j})$

$\Rightarrow\ a_1\hat{i} + a_2\hat{j} = (1 + 2\lambda)\hat{i} + (2 + 3\lambda)\hat{j}$

Comparing coefficients of $\hat{i}$ and $\hat{j}$, we get

$a_1 = 1 + 2\lambda,\ a_2 = 2 + 3\lambda$

$\therefore\ \vec{a} = (1 + 2\lambda)\hat{i} + (2 + 3\lambda)\hat{j}$

Now, $\vec{a}$ is perpendicular to $\hat{i} - \hat{j}$

$\therefore\ \vec{a}\cdot(\hat{i} - \hat{j}) = 0$

$\Rightarrow\ (1 + 2\lambda) - (2 + 3\lambda) = 0$

$\Rightarrow\ -1 - \lambda = 0\ \Rightarrow\ \lambda = -1$

$\therefore\ \vec{a} = -\hat{i} - \hat{j}.$

$\therefore\ \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\pm(-\hat{i} - \hat{j})}{\sqrt{1+1}} = \mp\frac{\hat{i} + \hat{j}}{\sqrt{2}}$

29. (a) : Here, we have $\vec{a} = \hat{i} - \hat{k},\ \vec{b} = x\hat{i} + \hat{j} + (1 - x)\hat{k}$ and $\vec{c} = y\hat{i} + x\hat{j} + (1 + x - y)\hat{k}$

Now, $[\vec{a}\ \vec{b}\ c] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$

Applying $C_3 \to C_3 + C_1$, we get

$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ y & x & 1+x \end{vmatrix} = 1[(1 + x) - x] = 1$

Thus, $[\vec{a}\ \vec{b}\ \vec{c}]$ depends neither on x nor on y.

30. (b) : Given that $|\vec{a}\times\vec{b}|^2 + |\vec{a}\cdot\vec{b}|^2 = 144$ and $|\vec{a}| = 4$

But $|\vec{a}\times\vec{b}|^2 + |\vec{a}\cdot\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$

$\Rightarrow\ 144 = 16\cdot|\vec{b}|^2$

$\Rightarrow\ |\vec{b}|^2 = 9\ \Rightarrow\ |\vec{b}| = 3$

31. (a) : Let $\vec{\alpha} = 4\hat{i} - \hat{j} + 3\hat{k},\ \vec{\beta} = -2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{\gamma} = x\hat{i} + y\hat{j} + z\hat{k}$ be the required vector.

Given that $|\vec{\gamma}| = 12$

$\Rightarrow\ x^2 + y^2 + z^2 = 144$ ... (i)

Also given that

$\vec{\gamma}\cdot\vec{\alpha} = 0\ \Rightarrow\ (x\hat{i} + y\hat{j} + z\hat{k})\cdot(4\hat{i} - \hat{j} + 3\hat{k}) = 0$

$\Rightarrow\ 4x - y + 3z = 0$ ... (ii)

and $\vec{\gamma}\cdot\vec{\beta} = 0\ \Rightarrow\ (x\hat{i} + y\hat{j} + z\hat{k})\cdot(-2\hat{i} + \hat{j} - 2\hat{k}) = 0$

$\Rightarrow\ -2x + y - 2z = 0$ ... (iii)

From (ii) and (iii), we get

$\frac{x}{2-3} = \frac{y}{-6+8} = \frac{z}{4-2}$

i.e., $\frac{x}{-1} = \frac{y}{2} = \frac{z}{2} = \lambda$ (say)

$\therefore\ x = -\lambda,\ y = 2\lambda$ and $z = 2\lambda$

Putting these values in (i), we get

$\lambda^2 + 4\lambda^2 + 4\lambda^2 = 144$

$\Rightarrow\ 9\lambda^2 = 144\ \Rightarrow\ \lambda^2 = 16$

$\Rightarrow\ \lambda = \pm 4$

$\therefore\ x = -4,\ y = z = 8$ or $x = 4,\ y = z = -8$

Therefore, the required vectors are

$\vec{\gamma} = -4\hat{i} + 8\hat{j} + 8\hat{k}$ or $\vec{\gamma} = 4\hat{i} - 8\hat{j} - 8\hat{k}$

32. (b) : By Lagrange's identity, we have

$(\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$

$\Rightarrow\ |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$

$= 16 - 4 = 12$

Again, $|\vec{c}|^2 = (2\vec{a}\times\vec{b} - 3\vec{b})\cdot(2\vec{a}\times\vec{b} - 3\vec{b})$

$|\vec{c}|^2 = 4|\vec{a}\times\vec{b}|^2 - 6\vec{b}\cdot(\vec{a}\times\vec{b}) - 6\vec{b}\cdot(\vec{a}\times\vec{b}) + 9|\vec{b}|^2$

$= 4|\vec{a}\times\vec{b}|^2 + 9|\vec{b}|^2$ $(\because\ \vec{b}\cdot(\vec{a}\times\vec{b}) = [\vec{b}\ \vec{a}\ \vec{b}] = 0)$

$= (4 \times 12) + (9 \times 16) = 192$

$\therefore\ |\vec{c}| = \sqrt{192} = 8\sqrt{3}$

Now, $\vec{b}\cdot\vec{c} = \vec{b}\cdot(2\vec{a}\times\vec{b} - 3\vec{b})$

$= 0 - 3|\vec{b}|^2 = -3\times 16 = -48$

Let the angle between $\vec{b}$ and $\vec{c}$ be θ

$$\therefore \quad \cos\theta = \frac{\vec{b}\cdot\vec{c}}{|\vec{b}||\vec{c}|} = \frac{-48}{4\cdot 8\sqrt{3}} = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \cos\theta = \cos\frac{5\pi}{6} \qquad \therefore \quad \theta = \frac{5\pi}{6}$$

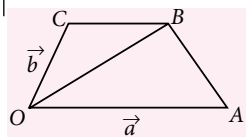
33. (c):

Here p = area of the quadrilateral $OABC$

$= \text{ar}(\Delta BOA) + \text{ar}(\Delta COB)$

$$= \left|\frac{1}{2}(\overrightarrow{OA}\times\overrightarrow{OB}) + \frac{1}{2}(\overrightarrow{OB}\times\overrightarrow{OC})\right|$$

$$= \frac{1}{2}|\overrightarrow{OA}\times\overrightarrow{OB} + \overrightarrow{OB}\times\overrightarrow{OC}|$$



$$= \frac{1}{2}|\vec{a}\times(10\vec{a}+2\vec{b}) + (10\vec{a}+2\vec{b})\times\vec{b}|$$

$$= \frac{1}{2}|10(\vec{a}\times\vec{a}) + 2(\vec{a}\times\vec{b}) + 10(\vec{a}\times\vec{b}) + 2(\vec{b}\times\vec{b})|$$

$$= \frac{1}{2}|0 + 12(\vec{a}\times\vec{b}) + 0| \quad [\because\ \vec{a}\times\vec{a} = \vec{b}\times\vec{b} = \vec{0}]$$

$$= 6|\vec{a}\times\vec{b}|$$

$\therefore \quad p = 6q \quad \Rightarrow \quad k = 6$

34. (b) : We know, three vectors will be coplanar, if their scalar triple product will be zero.

Now, $(2\hat{i}-\hat{j}+\hat{k})\times(\hat{i}+2\hat{j}-3\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix}$

$$= \hat{i} + 7\hat{j} + 5\hat{k}$$

$\therefore \quad (3\hat{i}+\lambda\hat{j}+5\hat{k})\cdot\{(2\hat{i}-\hat{j}+\hat{k})\times(\hat{i}+2\hat{j}-3\hat{k})\}$

$= (3\hat{i}+\lambda\hat{j}+5\hat{k})\cdot(\hat{i}+7\hat{j}+5\hat{k})$

$= 3 + 7\lambda + 25 = 7\lambda + 28$

$\therefore$ By the condition of coplanarity, we have

$7\lambda + 28 = 0 \quad \Rightarrow \quad \lambda = -4$

35. (b) : We have $[\vec{\alpha}+\vec{\beta} \quad \vec{\beta}+\vec{\gamma} \quad \vec{\gamma}+\vec{\alpha}]$

$= (\vec{\alpha}+\vec{\beta})\cdot\{(\vec{\beta}+\vec{\gamma})\times(\vec{\gamma}+\vec{\alpha})\}$

$= (\vec{\alpha}+\vec{\beta})\cdot(\vec{\beta}\times\vec{\gamma} + \vec{\beta}\times\vec{\alpha} + \vec{\gamma}\times\vec{\gamma} + \vec{\gamma}\times\vec{\alpha})$

$= (\vec{\alpha}+\vec{\beta})\cdot(\vec{\beta}\times\vec{\gamma} + \vec{\beta}\times\vec{\alpha} + \vec{\gamma}\times\vec{\alpha}) \quad [\because\ \vec{\gamma}\times\vec{\gamma} = \vec{0}]$

$= \vec{\alpha}\cdot(\vec{\beta}\times\vec{\gamma}) + \vec{\alpha}\cdot(\vec{\beta}\times\vec{\alpha}) + \vec{\alpha}\cdot(\vec{\gamma}\times\vec{\alpha}) + \vec{\beta}\cdot(\vec{\beta}\times\vec{\gamma})$

$\quad + \vec{\beta}\cdot(\vec{\beta}\times\vec{\alpha}) + \vec{\beta}\cdot(\vec{\gamma}\times\vec{\alpha})$

$= [\vec{\alpha} \quad \vec{\beta} \quad \vec{\gamma}] + 0 + 0 + 0 + 0 + [\vec{\beta} \quad \vec{\gamma} \quad \vec{\alpha}]$

$= [\vec{\alpha} \quad \vec{\beta} \quad \vec{\gamma}] + [\vec{\alpha} \quad \vec{\beta} \quad \vec{\gamma}] = 2[\vec{\alpha} \quad \vec{\beta} \quad \vec{\gamma}]$

$\therefore \quad \lambda = 2$

36. (a, b, c) : Let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

(a) $(\vec{a}\cdot\hat{i})\hat{i} + (\vec{a}\cdot\hat{j})\hat{j} + (\vec{a}\cdot\hat{k})\hat{k}$

$[\because\ \vec{a}\cdot\hat{i} = (x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{i} = x]$

$= x\hat{i} + y\hat{j} + z\hat{k} = \vec{a}$

(b) $(\vec{a}\cdot\hat{i})^2 + (\vec{a}\cdot\hat{j})^2 + (\vec{a}\cdot\hat{k})^2 = x^2 + y^2 + z^2 = |\vec{a}|^2$

(c) $\hat{i}\times(\vec{a}\times\hat{i}) + \hat{j}\times(\vec{a}\times\hat{j}) + \hat{k}\times(\vec{a}\times\hat{k})$

$= (\hat{i}\cdot\hat{i})\vec{a} - (\hat{i}\cdot\vec{a})\hat{i} + (\hat{j}\cdot\hat{j})\vec{a} - (\hat{j}\cdot\vec{a})\hat{j} + (\hat{k}\cdot\hat{k})\vec{a} - (\hat{k}\cdot\vec{a})\hat{k}$

$= \vec{a} - (\hat{i}\cdot\vec{a})\hat{i} + \vec{a} - (\hat{j}\cdot\vec{a})\hat{j} + \vec{a} - (\hat{k}\cdot\vec{a})\hat{k}$

$= 3\vec{a} - (x\hat{i} + y\hat{j} + z\hat{k}) = 3\vec{a} - \vec{a} = 2\vec{a}$

37. (a, b, c) : Here, $\vec{a} = \lambda\hat{i} + \hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} + \lambda\hat{j} - \hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + \lambda\hat{k}$ are coplanar.

$$\therefore \quad \vec{a}\cdot(\vec{b}\times\vec{c}) = 0 \quad \Rightarrow \quad \begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda(\lambda^2 - 1) - 1(\lambda + 2) + 2(-1 - 2\lambda) = 0$

$\Rightarrow \lambda^3 - \lambda - \lambda - 2 - 2 - 4\lambda = 0$

$\Rightarrow \lambda^3 - 6\lambda - 4 = 0$

$\Rightarrow \lambda^3 + 2\lambda^2 - 2\lambda^2 - 4\lambda - 2\lambda - 4 = 0$

$\Rightarrow \lambda^2(\lambda + 2) - 2\lambda(\lambda + 2) - 2(\lambda + 2) = 0$

$\Rightarrow (\lambda + 2)(\lambda^2 - 2\lambda - 2) = 0$

$\therefore$ either $\lambda + 2 = 0 \quad \Rightarrow \quad \lambda = -2$

or $\lambda^2 - 2\lambda - 2 = 0$

$$\Rightarrow \quad \lambda = \frac{2 \pm \sqrt{4 - 4\cdot 1(-2)}}{2\cdot 1} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

$\therefore$ The values of λ are -2, $1+\sqrt{3}$ and $1 \ -\sqrt{3}$.

38. (a, b, c, d) : $(\vec{a}-\vec{b})\times(\vec{b}-\vec{c})$

$= \vec{a}\times\vec{b} - \vec{a}\times\vec{c} - \vec{b}\times\vec{b} + \vec{b}\times\vec{c}$

$= \vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} \quad [\because\ \vec{b}\times\vec{b} = \vec{0} \text{ and } \vec{a}\times\vec{c} = -\vec{c}\times\vec{a}]$

$\therefore \quad (\vec{a}-\vec{b})\times(\vec{b}-\vec{c}) = \vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}$

Similarly, it can be shown that

$(\vec{c}-\vec{b})\times(\vec{a}-\vec{c})$ and $(\vec{a}-\vec{b})\times(\vec{a}-\vec{c})$

each is equal to $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$

Now, $\frac{1}{2}\{\vec{a}\times(\vec{b}-\vec{c})+\vec{b}\times(\vec{c}-\vec{a})+\vec{c}\times(\vec{a}-\vec{b})\}$

$=\frac{1}{2}(\vec{a}\times\vec{b}-\vec{a}\times\vec{c}+\vec{b}\times\vec{c}-\vec{b}\times\vec{a}+\vec{c}\times\vec{a}-\vec{c}\times\vec{b})$

$=\frac{1}{2}[(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})-(\vec{a}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{b})]$

$=\frac{1}{2}[2(\vec{a}\times\vec{b})+2(\vec{b}\times\vec{c})+2(\vec{c}\times\vec{a})]$

$=\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$

$\therefore\ \frac{1}{2}\{\vec{a}\times(\vec{b}-\vec{c})+\vec{b}\times(\vec{c}-\vec{a})+\vec{c}\times(\vec{a}-\vec{b})\}$

$=\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$

39. (a, b, c) : $[\vec{\alpha}\times\vec{a}\quad\vec{\beta}\times\vec{b}\quad\vec{\gamma}\times\vec{c}]$

Let $\vec{A}=\vec{\alpha}\times\vec{a},\ \vec{B}=\vec{\beta}\times\vec{b},\ \vec{C}=\vec{\gamma}\times\vec{c}$

So, $[\vec{A}\ \vec{B}\ \vec{C}]=\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}\cdot(\vec{A}\times\vec{B})$

Now,

$[\vec{A}\ \vec{B}\ \vec{C}]=\vec{A}\cdot(\vec{B}\times\vec{C})=(\vec{\alpha}\times\vec{a})\cdot[(\vec{\beta}\times\vec{b})\times(\vec{\gamma}\times\vec{c})]$

$=(\vec{\alpha}\times\vec{a})\cdot[\{(\vec{\beta}\times\vec{b})\cdot\vec{c}\}\vec{\gamma}-\{(\vec{\beta}\times\vec{b})\cdot\vec{\gamma}\}\vec{c}]$

$[\vec{A}\ \vec{B}\ \vec{C}]=[\vec{\beta}\ \vec{b}\ \vec{c}][\vec{\alpha}\ \vec{a}\ \vec{\gamma}]-[\vec{\beta}\ \vec{b}\ \vec{\gamma}][\vec{\alpha}\ \vec{a}\ \vec{c}]$

$\Rightarrow\ [\vec{\alpha}\times\vec{a}\quad\vec{\beta}\times\vec{b}\quad\vec{\gamma}\times\vec{c}]=[\vec{\alpha}\ \vec{a}\ \vec{\gamma}][\vec{c}\ \vec{\beta}\ \vec{b}]$

$-[\vec{\alpha}\ \vec{a}\ \vec{c}][\vec{\gamma}\ \vec{\beta}\ \vec{b}]$

Again, $[\vec{A}\ \vec{B}\ \vec{C}]=\vec{B}\cdot(\vec{C}\times\vec{A})=(\vec{\beta}\times\vec{b})\cdot[(\vec{\gamma}\times\vec{c})\times(\vec{\alpha}\times\vec{a})]$

$=(\vec{\beta}\times\vec{b})\cdot[\{(\vec{\gamma}\times\vec{c})\cdot\vec{a}\}\vec{\alpha}-\{(\vec{\gamma}\times\vec{c})\cdot\vec{\alpha}\}\vec{a}]$

$[\vec{A}\ \vec{B}\ \vec{C}]=[\vec{\gamma}\ \vec{c}\ \vec{a}][\vec{\beta}\ \vec{b}\ \vec{\alpha}]-[\vec{\gamma}\ \vec{c}\ \vec{\alpha}][\vec{\beta}\ \vec{b}\ \vec{a}]$

$\Rightarrow\ [\vec{\alpha}\times\vec{a}\quad\vec{\beta}\times\vec{b}\quad\vec{\gamma}\times\vec{c}]=[\vec{\gamma}\ \vec{c}\ \vec{a}][\vec{\beta}\ \vec{b}\ \vec{\alpha}]$

$-[\vec{\gamma}\ \vec{c}\ \vec{\alpha}][\vec{\beta}\ \vec{b}\ \vec{a}]$

Similarly, $[\vec{A}\ \vec{B}\ \vec{C}]=\vec{C}\cdot(\vec{A}\times\vec{B})$

$\Rightarrow\ [\vec{\alpha}\times\vec{a}\quad\vec{\beta}\times\vec{b}\quad\vec{\gamma}\times\vec{c}]=[\vec{\alpha}\ \vec{a}\ \vec{b}][\vec{\beta}\ \vec{\gamma}\ \vec{c}]$

$-[\vec{\alpha}\ \vec{a}\ \vec{\beta}][\vec{b}\ \vec{\gamma}\ \vec{c}]$

40. (a, d) : Let $\vec{c}=x\hat{i}+y\hat{j}+z\hat{k}$

Given that $|\vec{a}|=|\vec{b}|=|\vec{c}|=\sqrt{2}$

$\therefore\ x^2+y^2+z^2=2$ ... (i)

Also, $\cos\theta=\frac{\vec{a}\cdot\vec{b}}{\sqrt{2}\,\sqrt{2}}=\frac{\vec{b}\cdot\vec{c}}{\sqrt{2}\,\sqrt{2}}=\frac{\vec{c}\cdot\vec{a}}{\sqrt{2}\,\sqrt{2}}$

$\Rightarrow\ \frac{1}{2}=\frac{y+z}{2}=\frac{x+y}{2}$

$\therefore\ x+y=1,\ y+z=1$

Subtracting, $x-z=0$

$\Rightarrow\ x=z=1-y$

Putting the values of x and z in (i), we get

$(1-y)^2+y^2+(1-y)^2=2$

$\Rightarrow\ 3y^2-4y=0\ \Rightarrow\ y(3y-4)=0$

$\therefore\ y=0$ or $4/3$

When $y=0, x=z=1\ \Rightarrow\ \vec{c}=\hat{i}+\hat{k}$

When $y=\frac{4}{3}, x=z=-\frac{1}{3}$

$\Rightarrow\ \vec{c}=\frac{1}{3}(-\hat{i}+4\hat{j}-\hat{k})$

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$=\frac{1}{2}[(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})-(\vec{a}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{b})]$

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$=\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$

$\therefore\ \frac{1}{2}\{\vec{a}\times(\vec{b}-\vec{c})+\vec{b}\times(\vec{c}-\vec{a})+\vec{c}\times(\vec{a}-\vec{b})\}$

$=\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$

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$=(\vec{\alpha}\times\vec{a})\cdot[\{(\vec{\beta}\times\vec{b})\cdot\vec{c}\}\vec{\gamma}-\{(\vec{\beta}\times\vec{b})\cdot\vec{\gamma}\}\vec{c}]$

$[\vec{A}\ \vec{B}\ \vec{C}]=[\vec{\beta}\ \vec{b}\ \vec{c}][\vec{\alpha}\ \vec{a}\ \vec{\gamma}]-[\vec{\beta}\ \vec{b}\ \vec{\gamma}][\vec{\alpha}\ \vec{a}\ \vec{c}]$

$\Rightarrow\ [\vec{\alpha}\times\vec{a}\quad\vec{\beta}\times\vec{b}\quad\vec{\gamma}\times\vec{c}]=[\vec{\alpha}\ \vec{a}\ \vec{\gamma}][\vec{c}\ \vec{\beta}\ \vec{b}]$

$-[\vec{\alpha}\ \vec{a}\ \vec{c}][\vec{\gamma}\ \vec{\beta}\ \vec{b}]$

Again, $[\vec{A}\ \vec{B}\ \vec{C}]=\vec{B}\cdot(\vec{C}\times\vec{A})=(\vec{\beta}\times\vec{b})\cdot[(\vec{\gamma}\times\vec{c})\times(\vec{\alpha}\times\vec{a})]$

$=(\vec{\beta}\times\vec{b})\cdot[\{(\vec{\gamma}\times\vec{c})\cdot\vec{a}\}\vec{\alpha}-\{(\vec{\gamma}\times\vec{c})\cdot\vec{\alpha}\}\vec{a}]$

$[\vec{A}\ \vec{B}\ \vec{C}]=[\vec{\gamma}\ \vec{c}\ \vec{a}][\vec{\beta}\ \vec{b}\ \vec{\alpha}]-[\vec{\gamma}\ \vec{c}\ \vec{\alpha}][\vec{\beta}\ \vec{b}\ \vec{a}]$

$\Rightarrow\ [\vec{\alpha}\times\vec{a}\quad\vec{\beta}\times\vec{b}\quad\vec{\gamma}\times\vec{c}]=[\vec{\gamma}\ \vec{c}\ \vec{a}][\vec{\beta}\ \vec{b}\ \vec{\alpha}]$

$-[\vec{\gamma}\ \vec{c}\ \vec{\alpha}][\vec{\beta}\ \vec{b}\ \vec{a}]$

Similarly, $[\vec{A}\ \vec{B}\ \vec{C}]=\vec{C}\cdot(\vec{A}\times\vec{B})$

$\Rightarrow\ [\vec{\alpha}\times\vec{a}\quad\vec{\beta}\times\vec{b}\quad\vec{\gamma}\times\vec{c}]=[\vec{\alpha}\ \vec{a}\ \vec{b}][\vec{\beta}\ \vec{\gamma}\ \vec{c}]$

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$\therefore\ x+y=1,\ y+z=1$

Subtracting, $x-z=0$

$\Rightarrow\ x=z=1-y$

Putting the values of x and z in (i), we get

$(1-y)^2+y^2+(1-y)^2=2$

$\Rightarrow\ 3y^2-4y=0\ \Rightarrow\ y(3y-4)=0$

$\therefore\ y=0$ or $4/3$

When $y=0, x=z=1\ \Rightarrow\ \vec{c}=\hat{i}+\hat{k}$

When $y=\frac{4}{3}, x=z=-\frac{1}{3}$

$\Rightarrow\ \vec{c}=\frac{1}{3}(-\hat{i}+4\hat{j}-\hat{k})$

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INDEFINITE INTEGRALS

DEFINITION

Let $f(x)$ be a function. Then the family of all its primitives (or antiderivatives) is called the indefinite integral of $f(x)$ and is denoted by $\int f(x)dx$.

The symbol $\int f(x)dx$ is read as the indefinite integral of $f(x)$ with respect to x.

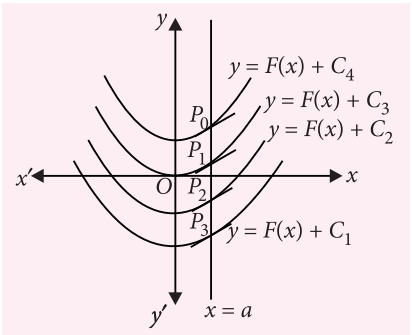
Thus, $\frac{d}{dx}(\phi(x)+C)=f(x)\Leftrightarrow\int f(x)dx=\phi(x)+C$,

where $\phi(x)$ is primitive of $f(x)$ and C is an arbitrary constant known as the constant of integration.

Here, $\int$ is integral sign, $f(x)$ is the integrand, x is the variable of integration and dx is the element of integration or differential of x.

GEOMETRICAL INTERPRETATION OF INDEFINITE INTEGRAL

The statement $\int f(x)dx=F(x)+C=y$ (say) represents a family of curves. The different values of C will correspond to different members of this family. Hence, the indefinite integral of a function represents geometrically, a family of curves placed parallel to each other having parallel tangents at the points of intersection of the curves of the family with the lines perpendicular to the axis representing the element of integration.



SOME FUNDAMENTAL INTEGRATION FORMULAS

(i) $\int x^n dx=\frac{x^{n+1}}{n+1}+C, n\neq -1$

(ii) $\int\frac{1}{x}dx=\log|x|+C$

(iii) $\int e^x dx=e^x+C$

(iv) $\int a^x dx=\frac{a^x}{\log a}+C$

(v) $\int\sin x\ dx=-\cos x+C$

(vi) $\int\cos x\ dx=\sin x+C$

(vii) $\int\sec^2 x\ dx=\tan x+C$

(viii) $\int\text{cosec}^2 x\ dx=-\cot x+C$

(ix) $\int\sec x\tan x\ dx=\sec x+C$

(x) $\int\text{cosec}\,x\cot x\ dx=-\text{cosec}\,x+C$

(xi) $\int\frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+C$

(xii) $\int-\frac{1}{\sqrt{a^2-x^2}}dx=\cos^{-1}\left(\frac{x}{a}\right)+C$

(xiii) $\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

(xiv) $\int-\frac{1}{a^2+x^2}dx=\frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right)+C$

(xv) $\int\frac{1}{x\sqrt{x^2-a^2}}dx=\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+C$

(xvi) $\int-\frac{1}{x\sqrt{x^2-a^2}}dx=\frac{1}{a}\text{cosec}^{-1}\left(\frac{x}{a}\right)+C$

SOME PROPERTIES OF INDEFINITE INTEGRAL

(a) The process of differentiation and integration are inverse of each other *i.e.*, $\frac{d}{dx}\int f(x)dx = f(x)$ and $\int f'(x)dx = f(x)+C$, where C is any arbitrary constant.

(b) Two indefinite integrals with the same derivative lead to the same family of curves so they are equivalent. So, if f and g are two functions such that $\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\int g(x)\,dx$, then $\int f(x)\,dx$ and $\int g(x)\,dx$ are equivalent.

(c) If c is any constant and $f(x)$ is a function of x, then $\int c\,f(x)dx = c\int f(x)dx$.

(d) The integral of the sum or the difference of two functions is equal to the sum or difference of their integrals *i.e.*,
$\int [f_1(x) \pm f_2(x)]dx = \int f_1(x)dx \pm \int f_2(x)dx$
Similarly,
$\int [f_1(x) \pm f_2(x) \pm ... \pm f_n(x)]dx = \int f_1(x)dx \pm \int f_2(x)dx \pm ... \pm \int f_n(x)dx$

METHODS OF INTEGRATION

(a) Integration by substitution
(b) Integration using partial fractions
(c) Integration by parts

(a) Integration by Substitution

By suitable substitution, the variable x in $\int f(x)dx$ is changed into another variable t so that the integrand $f(x)$ is changed in to $F(t)$, which is some integral or algebraic sum of standard integrals.

Some Important Substitutions

	Integrals	Substitutions
(i)	$\int f(ax+b)dx$	Put $ax + b = t$
(ii)	$\int x^{n-1} f(x^n)dx$	Put $x^n = t$
(iii)	$\int \{f(x)\}^n f'(x)dx$ or $\int \frac{f'(x)}{f(x)}\,dx$	Put $f(x) = t$
(iv)	$\int \frac{dx}{ax^2+bx+c}$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$	Transformed into standard form by expressing $ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 + \left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$ then put $x+\frac{b}{2a}=t$
(v)	$\int (ax+b)\sqrt{cx+d}\,dx$ or $\int \frac{ax+b}{\sqrt{cx+d}}dx$	Put $ax + b = A(cx + d) + B$
(vi)	$\int \frac{dx+e}{ax^2+bx+c}dx$ or $\int \frac{dx+e}{\sqrt{ax^2+bx+c}}dx$ or $\int (dx+e)\sqrt{ax^2+bx+c}\,dx$	Put $(dx+e) = A\frac{d}{dx}(ax^2+bx+c)+B$
(vii)	$\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}\,dx$	Put $a\sin x + b\cos x = A\frac{d}{dx}(c\sin x+d\cos x) + B(c\sin x+d\cos x)$

(b) Integration by Partial Fractions

If $p(x)$ and $q(x)$ are two polynomials such that the degree of $p(x)$ is less than the degree of $q(x)$, then we can evaluate $\int \frac{p(x)}{q(x)}dx$ by decomposing $\frac{p(x)}{q(x)}$ into partial fractions.

Method : First resolve the denominator of the given fraction into simplest factors. On the basis of these factors, we obtain the corresponding partial fraction as per rules given below:

	Factor in the denominator	Corresponding partial fraction
(i)	$(x - a)(x - b)$	$\frac{A}{x-a}+\frac{B}{x-b}$
(ii)	$(x - b)^2$	$\frac{A}{(x-b)}+\frac{B}{(x-b)^2}$
(iii)	$(x - c)^3$	$\frac{A}{(x-c)}+\frac{B}{(x-c)^2}+\frac{C}{(x-c)^3}$
(iv)	$(x - a)^2(x - b)$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
(v)	$(ax^2 + bx + c)$	$\frac{Ax+B}{(ax^2+bx+c)}$
(vi)	$(x - a)(bx^2 + cx + d)$ where $bx^2 + cx + d$ cannot be factorised further	$\frac{A}{x-a}+\frac{Bx+C}{bx^2+cx+d}$

(c) Integration by Parts

The process of integration of the product of two functions is known as integration by parts.

If u and v are two functions of x, then

$$\int uvdx = u\cdot(\int vdx) - \int\left\{\frac{du}{dx}\cdot\int vdx\right\}dx$$

In words, integral of the product of two functions = first function × integral of the second – integral of (differential coefficient of the first function × integral of the second function).

Note : We choose the first function as the function which comes first in the word ILATE, where

I–stands for the inverse trigonometric function ($\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc.)

L – stands for the logarithmic functions

A – stands for the algebraic functions

T – stands for the trigonometric functions

E – stands for the exponential functions

If there is no other function, then unity is taken as the second function.

Some Particular Integrals

(i) $\int \cot x\ dx = \log|\sin x| + C = -\log \operatorname{cosec} x + C$

(ii) $\int \tan x\ dx = -\log|\cos x| + C = \log|\sec x| + C$

(iii) $\int \sec x\ dx = \log|\sec x + \tan x| + C$

(iv) $\int \operatorname{cosec} x\ dx = \log|\operatorname{cosec} x - \cot x| + C$

(v) $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$

(vi) $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$

(vii) $\int\frac{dx}{\sqrt{x^2+a^2}}=\log\left|x+\sqrt{x^2+a^2}\right|+C$

(viii) $\int\frac{dx}{\sqrt{x^2-a^2}}=\log\left|x+\sqrt{x^2-a^2}\right|+C$

(ix) $\int\sqrt{a^2+x^2}dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right|+C$

(x) $\int\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

(xi) $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C$

Special Integrals

(i) $\int e^x[f(x) + f'(x)]dx = e^x f(x) + C$

(ii) $\int e^{kx}[kf(x) + f'(x)]dx = e^{kx} f(x) + C$

DEFINITE INTEGRALS

DEFINITION

Let $F(x)$ be antiderivative of $f(x)$, then for any two values of the independent variable x, say a and b, the difference $F(b) - F(a)$ is called the definite integral of $f(x)$ from a to b and is denoted by $\int_a^b f(x)dx$. Thus

$$\int_a^b f(x)dx = F(b) - F(a)$$

where numbers a and b are called the limits of integration; a is the lower limit and b is the upper limit.

DEFINITE INTEGRAL AS THE LIMIT OF A SUM

Let f be a continuous function defined in a closed interval $[a, b]$, then the definite integral $\int_a^b f(x)dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis.

Also,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+...+f\{a+(n-1)h\}\right]$$

where, $h=\frac{b-a}{n}, h\to 0$ as $n\to\infty$

FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

(i) Area function : The function $A(x)$ denotes the area function and is given by $A(x)=\int_a^x f(x)dx$.

(ii) First Fundamental Theorem : Let $f(x)$ be a continuous function defined in the closed interval $[a, b]$ and $A(x)$ be the area function, then $A'(x) = f(x)$, for all $x \in [a, b]$.

(iii) Second Fundamental Theorem : Let $f(x)$ be a continuous function defined in the closed interval $[a, b]$ and $\int f(x)\,dx = F(x)$, then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b)-F(a).$$

PROPERTIES OF DEFINITE INTEGRALS

(i) $\int_a^b f(x)dx = \int_a^b f(y)dy$

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(iii) $\int_a^a f(x)dx = 0$

(iv) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a<c<b$

(v) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

(vi) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

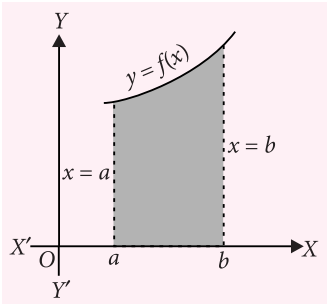
(vii) $\int_{-a}^a f(x)dx = \begin{cases} 0, & \text{if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function} \end{cases}$

(viii) $\int_0^{2a} f(x)dx = \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \end{cases}$

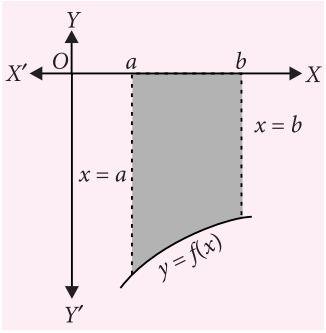
APPLICATION OF INTEGRALS

1. Area of a curve between two ordinates : Let $y = f(x)$ be a continuous and finite function in $[a, b]$.

Case I : If the curve $y = f(x)$ lies above the x-axis, then the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is given by $\int_a^b y\,dx$

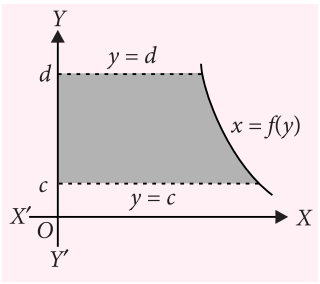


Case II : If the curve $y = f(x)$ lies below the x-axis, then the area between the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is given by $\int_a^b (-y)\,dx$

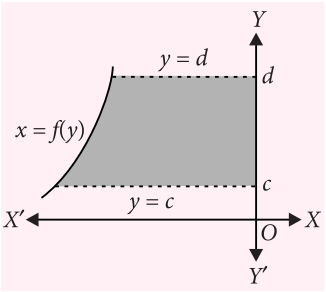


2. Area of a curve between two abscissas :

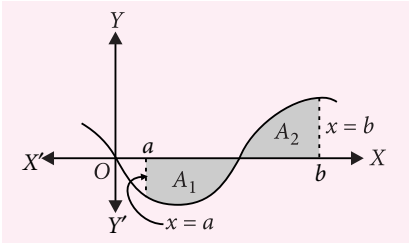
Case I : If the curve $x = f(y)$ lies to the right of the y-axis, then the area bounded by the curve $x = f(y)$, y-axis and the abscissa $y = c$ and $y = d$ is given by $\int_c^d x\,dy$



Case II : If the curve $x = f(y)$ lies to the left of the y-axis, then the area bounded by the curve $x = f(y)$, y-axis and the abscissa $y = c$ and $y = d$ is given by $\int_c^d (-x)\,dy$



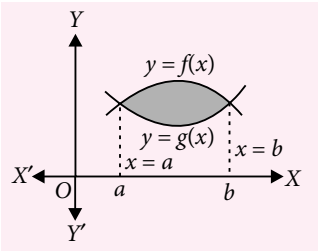
Note : If some portion of the curve is above the x-axis and some is below the x-axis as shown in the figure, then $A_1 < 0$ and $A_2 > 0$. Therefore, the area A bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is given by $A = |A_1| + A_2$.



3. Area between two curves :

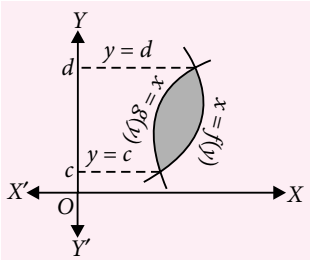
Case I : The area bounded by two curves $y = f(x)$ and $y = g(x)$, which intersects at the ordinates $x = a$ and $x = b$, is given by

$$\int_a^b |f(x) - g(x)|\,dx$$



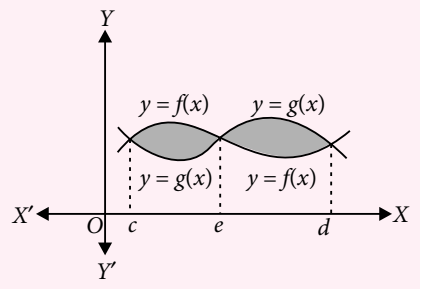
Case II : The area bounded by two curves $x = f(y)$ and $x = g(y)$ which intersects at the abscissas, $y = c$ and $y = d$ is given by

$$\int_c^d |f(y) - g(y)|\,dy$$



Note : If $f(x) \geq g(x)$ in $[c, e]$ and $f(x) \leq g(x)$ in $[e, d]$ where $c < e < d$, then area bounded by the curves is,

$$\int_c^e [f(x) - g(x)]\,dx + \int_e^d [g(x) - f(x)]\,dx$$



DIFFERENTIAL EQUATION

DEFINITION

An equation involving an independent variable, dependent variable and the derivatives of the dependent variable, is called a differential equation.

- A differential equation involving derivatives of the dependent variable with respect to only one independent variable is called an **ordinary differential equation** and a differential equation involving derivatives with respect to more than one independent variables is called a **partial differential equation.**

ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

The order of highest order derivative appearing in a differential equation is called the order of the differential equation.

The power of the highest order derivative appearing in a differential equation, after it is made free from radicals and fraction, is called the degree of the differential equation.

Note :

- The degree of a differential equation which is not a polynomial equation in derivatives is not defined.
- Order and degree (if defined) of a differential equation are always positive integers.

SOLUTION OF A DIFFERENTIAL EQUATION

Any relation between the dependent and independent variables (not involving the derivatives) which, satisfy the given differential equation is called a solution of the differential equation.

General Solution : The solution which contains arbitrary constants is called the general solution (primitive) of the differential equation.

Particular Solution : The solution obtained from the general solution by giving particular value to arbitrary constants is called a particular solution.

FORMATION OF A DIFFERENTIAL EQUATION WHOSE GENERAL SOLUTION IS GIVEN

Suppose an equation of a family of curves contains n arbitrary constants (called parameters).

Then, we obtain its differential equation, as given below.

Step I : Differentiate the equation of the given family of curves n times to get n more equations.

Step II : Eliminate n constants, using these $(n + 1)$ equations.

This gives us the required differential equation of order n.

METHODS OF SOLVING FIRST ORDER, FIRST DEGREE DIFFERENTIAL EQUATIONS

(i) If the equation is $\frac{dy}{dx} = f(x)$, then $y = \int f(x)dx + C$ is the solution.

(ii) Variable separable : If the given differential equation can be expressed in the form $f(x)dx = g(y)dy$, then $\int f(x)dx = \int g(y)dy + C$ is the solution.

(iii) Reducible to variable separable : If the equation is $\frac{dy}{dx} = f(ax+by+c)$, then put $ax + by + c = z$.

(iv) Homogeneous equation : If a first order, first degree differential equation is expressible in the form

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)},$$

where $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree in x and y, then put $y = vx$

(v) Linear Equation : If the equation is $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x, then

$$y \cdot e^{\int Pdx} = \int Q \cdot e^{\int Pdx} dx + C,$$

where $e^{\int Pdx}$ is the integrating factor (I.F.).

OR

If the equation is $\frac{dx}{dy} + Px = Q$, where P and Q are functions of y, then

$$x \cdot e^{\int Pdy} = Q \cdot e^{\int Pdy} dy + C,$$

where $e^{\int Pdy}$ is the integrating factor.

VERY SHORT ANSWER TYPE

1. Evaluate : $\int 2^{(x+3)}dx$

2. Evaluate : $\int_0^{\pi/2} x\cos x\,dx$

3. Find the differential equation of the family of curves $y = Ae^x + Be^{-x}$, where A and B are arbitrary constants.

4. Determine the order and degree of the differential equation $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0$

5. Solve : $\frac{dy}{dx} = e^{x+y}$

6. Evaluate : $\int \cos^3 x \sin x\,dx$

SHORT ANSWER TYPE

7. For what values of c and a is the following equation satisfied?

$$\int (\sin 2x + \cos 2x)dx = \frac{1}{\sqrt{2}}\sin(2x - c) + a$$

8. Evaluate : $\int \frac{\log x - 1}{(\log x)^2} dx$

9. If the area enclosed between the curves $y = ax^2$ and $x = ay^2$ $(a > 0)$ is 1 square unit, then find the value of a.

10. The line normal to a given curve at each point (x, y) on the curve passes through the point (3, 0). If the curve contains the point (3, 4), find its equation.

11. Evaluate : $\int_0^{\pi/2} \frac{\sin^2 x}{(1+\sin x\cos x)} dx$

12. Solve : $x\frac{dy}{dx} - y = \log x$

LONG ANSWER TYPE

13. Evaluate : $\int \frac{x}{\sqrt{8+x-x^2}} dx$

14. Evaluate $\int_1^3 (x^2 + x)dx$ as limit of sum.

15. Evaluate : $\int \frac{x^4}{(x-1)(x^2+1)} dx$

16. Solve :

$$\left(x\cos\frac{y}{x}\right)(y\,dx + x\,dy) = \left(y\sin\frac{y}{x}\right)(x\,dy - y\,dx)$$

17. Find the area of the region $\{(x, y) : x^2 + y^2 \le 2ax, y^2 \ge ax, x \ge 0, y \ge 0\}$.

18. Evaluate the following integrals :

(i) $\int \frac{x^2+1}{\sqrt{x^2+4}} dx$ **(ii)** $\int \frac{x^2+9}{x^4-2x^2+81} dx$

SOLUTIONS

1. $\int 2^{(x+3)}dx = \int 2^x \cdot 2^3 dx = 8\int 2^x dx$

$$= 8\cdot\frac{2^x}{\log 2} + C \quad = \frac{2^{(x+3)}}{\log 2} + C$$

2. $\int_0^{\pi/2} x\cos x\,dx = [x\sin x]_0^{\pi/2} - \int_0^{\pi/2} 1\cdot\sin x\,dx$

[Integrating by parts]

$$= \frac{\pi}{2} + [\cos x]_0^{\pi/2} = \frac{\pi}{2} - 1$$

3. Given equation is

$$y = Ae^x + Be^{-x} \quad ...(i)$$

Now, $\frac{dy}{dx} = Ae^x - Be^{-x}$

$\Rightarrow \quad \frac{d^2y}{dx^2} = Ae^x + Be^{-x} = y$ [From (i)]

$\Rightarrow \quad \frac{d^2y}{dx^2} - y = 0$ is required differential equation.

4. The highest order derivative present in the differential equation is $\frac{dy}{dx}$. So, it is of order 1.

Also, L.H.S. of the differential equation cannot be expressed as a polynomial in $\frac{dy}{dx}$. So, its degree is not defined.

5. We have, $\frac{dy}{dx} = e^{x+y}$

$\Rightarrow \quad \frac{dy}{dx} = e^x e^y \Rightarrow \quad e^{-y}\,dy = e^x\,dx$

$\Rightarrow \quad \int e^{-y}\,dy = \int e^x\,dx$

$\Rightarrow \quad -e^{-y} = e^x + C$, which is the required solution.

6. We have, $\int \cos^3 x \sin x\,dx$

Put $\cos x = t \Rightarrow \sin x\,dx = -dt$

$$\therefore \quad \int \cos^3 x \sin x\,dx = -\int t^3 dt = -\frac{t^4}{4} + C$$

$$= -\frac{1}{4}\cos^4 x + C$$

7. $\int (\sin 2x + \cos 2x)dx = -\frac{\cos 2x}{2} + \frac{\sin 2x}{2} + C$,

where C is an arbitrary constant.

$$= \frac{1}{2}(\sin 2x - \cos 2x) + C$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\sin 2x - \frac{1}{\sqrt{2}}\cos 2x\right) + C$$

$$= \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4}\sin 2x - \sin\frac{\pi}{4}\cos 2x\right) + C$$

$$= \frac{1}{\sqrt{2}}\sin\left(2x - \frac{\pi}{4}\right) + C$$

But $\int (\sin 2x + \cos 2x)dx = \frac{1}{\sqrt{2}}\sin(2x - c) + a$

$$\therefore \quad \frac{1}{\sqrt{2}}\sin\left(2x - \frac{\pi}{4}\right) + C = \frac{1}{\sqrt{2}}\sin(2x - c) + a$$

$\Rightarrow \quad c = \frac{\pi}{4}$ and $a = C$, an arbitrary constant.

8. Let $I = \int \frac{\log x - 1}{(\log x)^2}dx$

Put $\log x = t \quad \Rightarrow \quad x = e^t \quad \therefore \quad dx = e^t\,dt$

$$\therefore \quad I = \int \frac{t-1}{t^2}\cdot e^t dt = \int e^t\left(\frac{1}{t} - \frac{1}{t^2}\right)dt$$

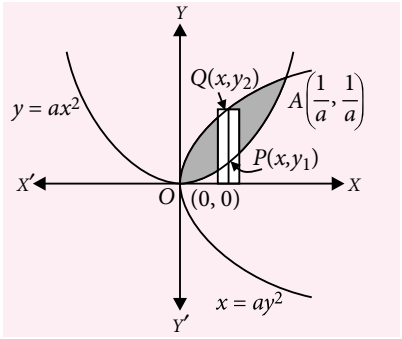
$$= \int \frac{1}{t}\cdot e^t dt - \int \frac{1}{t}\cdot e^t dt$$

Integrating first integral by parts, we get

$$I = \frac{1}{t}\cdot e^t - \int -\frac{1}{t^2}\cdot e^t dt - \int \frac{1}{t^2}\cdot e^t dt$$

$$= \frac{1}{t}\cdot e^t + C = \frac{x}{\log x} + C$$

9. Solving, $y = ax^2$ and $x = ay^2$, we get point of intersection $O(0, 0)$ and $A\left(\frac{1}{a}, \frac{1}{a}\right)$.



Since, area enclosed between the curves = 1

$$\Rightarrow \quad \int_0^{1/a}\left\{\sqrt{\frac{x}{a}} - ax^2\right\}dx = 1$$

$$\Rightarrow \quad \left[\frac{2}{3\sqrt{a}}x^{3/2} - \frac{a}{3}x^3\right]_0^{1/a} = 1$$

$$\Rightarrow \quad \frac{2}{3\sqrt{a}} \times \frac{1}{a^{3/2}} - \frac{a}{3} \times \frac{1}{a^3} = 1 \Rightarrow \quad \frac{2}{3a^2} - \frac{1}{3a^2} = 1$$

$$\Rightarrow \quad \frac{1}{3a^2} = 1 \quad \Rightarrow \quad a^2 = \frac{1}{3} \quad \Rightarrow \quad a = \frac{1}{\sqrt{3}} \quad [\because a > 0]$$

10. The equation of the normal to a curve at a point (x, y) is

$$(Y - y)\frac{dy}{dx} + (X - x) = 0$$

Since it passes through the point (3, 0), we have

$$(0 - y)\frac{dy}{dx} + (3 - x) = 0 \quad \Rightarrow \quad y\frac{dy}{dx} = (3 - x)$$

$$\Rightarrow \quad \int y\,dy = \int (3 - x)dx \quad \Rightarrow \quad \frac{y^2}{2} = 3x - \frac{x^2}{2} + C$$

$\Rightarrow \quad x^2 + y^2 - 6x - 2C = 0$,

Which is the equation of the curve.

Since the curve passes through (3, 4), we have

$9 + 16 - 18 - 2C = 0 \Rightarrow C = 7/2$

$\therefore\ x^2 + y^2 - 6x - 7 = 0$ is the required equation of the curve.

11. Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{(1+\sin x \cos x)}dx$ ...(i)

Then, $I = \int_0^{\pi/2} \frac{\sin^2[(\pi/2)-x]}{1+\sin[(\pi/2)-x]\cos[(\pi/2)-x]}dx$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

or $I = \int_0^{\pi/2} \frac{\cos^2 x}{(1+\sin x \cos x)}dx$ ...(ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{(\sin^2 x + \cos^2 x)}{(1+\sin x \cos x)}dx = \int_0^{\pi/2} \frac{dx}{(1+\sin x \cos x)}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{(\sec^2 x + \tan x)}dx$$

(By dividing numerator and denominator by $\cos^2 x$)

$$= \int_0^{\pi/2} \frac{\sec^2 x}{(1+\tan^2 x + \tan x)}dx = \int_0^{\infty} \frac{dt}{(t^2+t+1)},$$

[Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$,

When $x = 0 \Rightarrow t = 0$ and $x = \frac{\pi}{2} \Rightarrow t \to \infty$]

$$= \int_0^{\infty} \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{2}{\sqrt{3}}\cdot\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{2\pi}{3\sqrt{3}}$$

12. The given equation can be written as

$\frac{dy}{dx} - \frac{1}{x}\cdot y = \frac{\log x}{x}$

This is of the linear form $\frac{dy}{dx} + Py = Q$, where $P = \frac{-1}{x}$ and $Q = \frac{\log x}{x}$.

$$I.F. = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log(x^{-1})} = x^{-1} = \frac{1}{x}$$

So, the required solution is

$y \times I.F. = \int\{Q \times (I.F.)\}dx + C$

$$\Rightarrow \quad y \times \frac{1}{x} = \int\left(\frac{\log x}{x} \times \frac{1}{x}\right)dx + C$$

$$\Rightarrow \quad \frac{y}{x} = \int (\log x)\cdot\frac{1}{x^2}dx + C$$

$$\Rightarrow \quad \frac{y}{x} = (\log x)\left(-\frac{1}{x}\right) - \int\frac{1}{x}\cdot\left(-\frac{1}{x}\right)dx + C$$

[Integrating by parts]

$$\Rightarrow \quad \frac{y}{x} = -\frac{\log x}{x} + \int\frac{1}{x^2}dx + C = \frac{-\log x}{x} - \frac{1}{x} + C$$

$\Rightarrow\ y = Cx - (\log x + 1)$

13. Let $I = \int \frac{x}{\sqrt{8+x-x^2}}dx$

Let $x = A(1 - 2x) + B$ or $x = -2Ax + (B + A)$

Equating the coefficient of x and constant term, we get

$-2A = 1 \Rightarrow A = -\frac{1}{2}$ and $0 = B + A \Rightarrow B = -A = \frac{1}{2}$

$$\therefore \quad x = -\frac{1}{2}(1-2x) + \frac{1}{2}$$

$$\therefore \quad I = \int \frac{-\frac{1}{2}(1-2x)+\frac{1}{2}}{\sqrt{8+x-x^2}}dx$$

$$= -\frac{1}{2}\int\frac{1-2x}{\sqrt{8+x-x^2}}dx + \frac{1}{2}\int\frac{dx}{\sqrt{8+x-x^2}}$$

Let $I = -\frac{1}{2}I_1 + \frac{1}{2}I_2$, where

$I_1 = \int\frac{1-2x}{\sqrt{8+x-x^2}}$ and $I_2 = \int\frac{dx}{\sqrt{8+x-x^2}}$

Now, $I_1 = \int\frac{1-2x}{\sqrt{8+x-x^2}}dx$

Put $8 + x - x^2 = t \Rightarrow (1 - 2x)dx = dt$

$$\therefore \quad I_1 = \int\frac{dt}{t^{1/2}} = \int t^{-1/2}dt = 2\sqrt{t} = 2\sqrt{8+x-x^2}$$

Also, $I_2 = \int\frac{dx}{\sqrt{8+x-x^2}} = \int\frac{dx}{\sqrt{8-(x^2-x)}}$

$$= \int \frac{dx}{\sqrt{8-\left(x^2-2\times x\times\frac{1}{2}+\frac{1}{4}\right)+\frac{1}{4}}} = \int \frac{dx}{\sqrt{\frac{33}{4}-\left(x-\frac{1}{2}\right)^2}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{\sqrt{33}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}} = \sin^{-1}\left(\frac{x-1/2}{\sqrt{33}/2}\right)$$

$$= \sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right)$$

$$\therefore \quad I = -\frac{1}{2}\times 2\sqrt{8+x-x^2}+\frac{1}{2}\sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right)+C$$

$$= -\sqrt{8+x-x^2}+\frac{1}{2}\sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right)+C$$

14. Let $I = \int_1^3 (x^2+x)\,dx$

By definition, we know

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h) +\ldots+f(a+(n-1)h)],$$

where $h = \frac{b-a}{n}$

Here, $a = 1$, $b = 3$, $f(x) = x^2 + x$ and $h = \frac{3-1}{n} = \frac{2}{n}$

$$\Rightarrow I = \lim_{h\to 0} h[f(1)+f(1+h)+f(1+2h)+f(1+3h) +\ldots+f(1+(n-1)h)]$$

$$\Rightarrow I = \lim_{h\to 0} h[\{1^2+1\}+\{(1+h)^2+(1+h)\} +\{(1+2h)^2+(1+2h)\}+\ldots+\{(1+(n-1)h)^2 +(1+(n-1)h)\}]$$

$$\Rightarrow I = \lim_{h\to 0} h[\{1^2+(1+h)^2+(1+2h)^2+\ldots +\ldots+(1+(n-1)h)^2\}+\{1+(1+h)+(1+2h) +\ldots+(1+(n-1)h)\}]$$

$$\Rightarrow I = \lim_{h\to 0} h[\{n+2h(1+2+3+\ldots+(n-1)) +h^2(1^2+2^2+\ldots+(n-1)^2)\} +\{n+h(1+2+3+\ldots+(n-1))\}]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[n+\left(2h\times\frac{n(n-1)}{2}\right) +\left(h^2\times\frac{n(n-1)(2n-1)}{6}\right)+n+\left(h\times\frac{n(n-1)}{2}\right)\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[2n+\left(3h\times\frac{n(n-1)}{2}\right) +\left(h^2\times\frac{n(n-1)(2n-1)}{6}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[2n+\left(\frac{6}{n}\times\frac{n(n-1)}{2}\right) +\left(\frac{4}{n^2}\times\frac{n(n-1)(2n-1)}{6}\right)\right] \left[\because h=\frac{2}{n}\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[4+6\left(\frac{n-1}{n}\right)+\frac{4}{3}\frac{(n-1)(2n-1)}{n^2}\right]$$

$$= \lim_{n\to\infty}\left[4+6\left(1-\frac{1}{n}\right)+\frac{4}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right]$$

$$\Rightarrow I = 4+6(1-0)+\frac{4}{3}(1-0)(2-0) = 4+6+\frac{8}{3} = \frac{38}{3}$$

15. We have, $\int \frac{x^4}{(x-1)(x^2+1)}\,dx$

$$\because \quad \frac{x^4}{(x-1)(x^2+1)} = \frac{x^4}{(x^3-x^2+x-1)} = (x+1)+\frac{1}{(x^3-x^2+x-1)}$$

$$\Rightarrow \quad \frac{x^4}{(x-1)(x^2+1)} = (x+1)+\frac{1}{(x-1)(x^2+1)} \qquad \ldots(i)$$

Let $\frac{1}{(x-1)(x^2+1)} = \frac{A}{(x-1)}+\frac{Bx+C}{(x^2+1)}$

Then, $1 = A(x^2+1)+(Bx+C)(x-1)$ ...(ii)

Putting $x = 1$ in (ii), we get $A = \frac{1}{2}$

Comparing coefficients of x^2 on both sides of (ii), we get

$$A + B = 0 \Rightarrow B = -A = -\frac{1}{2}$$

Comparing the constant terms on both sides of (ii), we get

$$A - C = 1 \Rightarrow C = A-1 = \frac{-1}{2}$$

$$\therefore \quad \frac{1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)}+\frac{-\frac{1}{2}x-\frac{1}{2}}{(x^2+1)} \qquad \ldots(iii)$$

$$\therefore \quad \frac{x^4}{(x-1)(x^2+1)} = (x+1)+\frac{1}{2(x-1)}-\frac{1}{2}\frac{(x+1)}{(x^2+1)}$$

[By (i) & (iii)]

$$\Rightarrow \int \frac{x^4}{(x-1)(x^2+1)}dx = \int (x+1)dx + \frac{1}{2}\int \frac{dx}{(x-1)} - \frac{1}{4}\cdot\int \frac{2x}{(x^2+1)}dx - \frac{1}{2}\int \frac{dx}{(x^2+1)}$$

$$= \frac{x^2}{2} + x + \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + C$$

16. The given equation may be written as

$$\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y - \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{\{x\cos(y/x) + y\sin(y/x)\}y}{\{y\sin(y/x) - x\cos(y/x)\}x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\{\cos(y/x) + (y/x)\sin(y/x)\}(y/x)}{\{(y/x)\sin(y/x) - \cos(y/x)\}},$$

which is clearly homogeneous, being a function of (y/x).

Putting, $y = vx$ and $\frac{dy}{dx} = \left(v + x\frac{dv}{dx}\right)$ in it, we get

$$v + x\frac{dv}{dx} = \frac{v(\cos v + v\sin v)}{(v\sin v - \cos v)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{(v\sin v - \cos v)}$$

$$\Rightarrow \int \frac{(v\sin v - \cos v)}{v\cos v}dv = \int \frac{2}{x}dx$$

$$\Rightarrow \int \tan v\, dv - \int \frac{dv}{v} = \int \frac{2}{x}dx$$

$$\Rightarrow -\log|\cos v| - \log|v| + \log C = 2\log|x|$$

$$\Rightarrow \log|\cos v| + \log|v| + 2\log|x| = \log C$$

$$\Rightarrow \log|x^2 v\cos v| = \log C \Rightarrow |x^2 v\cos v| = C$$

$$\Rightarrow x^2 v\cos v = \pm C = C_1 \text{ (say)}$$

$$\Rightarrow xy\cos\left(\frac{y}{x}\right) = C_1, \text{ which is the required solution.}$$

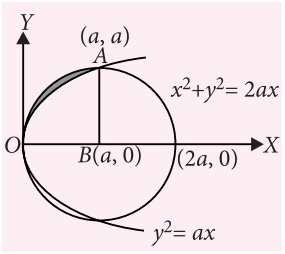
17. The equations of the given curves are

$x^2 + y^2 = 2ax$ ... (i)

and $y^2 = ax$ ... (ii)

Now,

clearly $x^2 + y^2 = 2ax$ is a circle with its centre $B(a, 0)$ and radius $= a$ units.



Since $y^2 = ax$ is a parabola with $O(0, 0)$ as its vertex and the x-axis as its axis, then

solving (i) and (ii), we get the points of intersection $O(0, 0)$ and $A(a, a)$.

$$\therefore \text{ Required area } = \int_0^a \sqrt{2ax - x^2}\ dx - \int_0^a \sqrt{ax}\ dx$$

$$= \int_0^a \sqrt{a^2 - (x-a)^2}\ dx - \sqrt{a}\int_0^a \sqrt{x}\ dx$$

$$= \left[\frac{(x-a)\sqrt{a^2-(x-a)^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right]_0^a - \sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^a$$

$$= \left\{\frac{a^2}{2}\sin^{-1}(0) - \frac{a^2}{2}\sin^{-1}(-1) - \frac{2}{3}a^2\right\}$$

$$= \left\{\frac{\pi a^2}{4} - \frac{2}{3}a^2\right\} \text{sq. units}$$

18. (i) $$\int \frac{x^2+1}{\sqrt{x^2+4}}dx = \int \frac{x^2+4-3}{\sqrt{x^2+4}}dx$$

$$= \int \frac{x^2+4}{\sqrt{x^2+4}}dx - 3\int \frac{dx}{\sqrt{x^2+2^2}}$$

$$= \int \sqrt{x^2+2^2}dx - 3\int \frac{dx}{\sqrt{x^2+2^2}}$$

$$= \frac{x}{2}\sqrt{x^2+2^2} + \frac{2^2}{2}\log\left|x + \sqrt{x^2+2^2}\right| - 3\log\left|x + \sqrt{x^2+4}\right| + C$$

$$= \frac{x}{2}\sqrt{x^2+4} + 2\log\left|x + \sqrt{x^2+4}\right| - 3\log\left|x + \sqrt{x^2+4}\right| + C$$

$$= \frac{x}{2}\sqrt{x^2+4} - \log\left|x + \sqrt{x^2+4}\right| + C$$

(ii) Let $I = \int \frac{x^2+9}{x^4 - 2x^2 + 81}dx$

$$= \int \frac{1 + \frac{9}{x^2}}{x^2 - 2 + \frac{81}{x^2}}dx = \int \frac{1 + \frac{9}{x^2}}{x^2 + \frac{81}{x^2} - 2}dx$$

$$= \int \frac{1 + \frac{9}{x^2}}{\left(x - \frac{9}{x}\right)^2 + 18 - 2}dx$$

Put $x - \frac{9}{x} = t \Rightarrow \left(1 + \frac{9}{x^2}\right)dx = dt$

$$\therefore \quad I = \int \frac{dt}{t^2+16} = \frac{1}{4}\tan^{-1}\left(\frac{t}{4}\right) + C$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{x - \frac{9}{x}}{4}\right) + C = \frac{1}{4}\tan^{-1}\left(\frac{x^2-9}{4x}\right) + C$$

Math Archives 10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE (Main & Advanced) Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE (Main & Advanced). In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. If [.] stands for the greatest integer function, $\int_1^2 [3x]\,dx$ is equal to

(a) 3 (b) 4 (c) 5 (d) 6

2. If $\int_{\ln 2}^{x} \frac{du}{\sqrt{e^u-1}} = \frac{\pi}{6}$, then the value of x is

(a) 4 (b) ln 8
(c) ln 4 (d) None of these

3. The equation of a curve is $y = f(x)$. The tangents at $(1, f(1))$, $(2, f(2))$ and $(3, f(3))$ make angles $\frac{\pi}{6}, \frac{\pi}{3}$ and $\frac{\pi}{4}$ respectively with positive direction of x-axis, then the value of $\int_2^3 f'(x)f''(x)dx + \int_1^3 f''(x)\,dx$ is equal to

(a) $-\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$
(c) 0 (d) None of these

4. Let $f(x)$ be a function defined by

$f(x) = \int_1^x t(t^2-3t+2)\,dt,\ 1 \le x \le 3.$

Then the range of $f(x)$ is

(a) [0, 2] (b) $\left[-\frac{1}{4}, 4\right]$
(c) $\left[-\frac{1}{4}, 2\right]$ (d) None of these

5. All the values of 'a' for which $\int_1^2 (a^2+(4-4a)x+4x^3)\,dx \le 12$ are given by

(a) $a = 3$ (b) $a \le 4$
(c) $0 \le a < 3$ (d) None of these

6. Evaluate $\int \frac{(1+x)\sin x}{(x^2+2x)\cos^2 x-(1+x)\sin 2x}\,dx$

7. Show that the value of $\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}\,dx,\ n \in N$, is independent of n. Find the value of the integral.

8. Find the orthogonal trajectories of the family of circles having their centres on the y-axis and touching the x-axis.

9. Find the area bounded by $x = \cos^{-1} y$, x-axis and the lines $|x| = 1$.

10. Show that the solution of the differential equation $yp^2 - p = (p+2)(py-1)$, $p \equiv dy/dx$, represents two families,one of which is the family of curves having subnormal unity and the other is the family of parallel lines.

SOLUTIONS

1. (b): $\int_1^2 [3x]\,dx = \int_1^{4/3} [3x]\,dx + \int_{4/3}^{5/3} [3x]\,dx + \int_{5/3}^{2} [3x]\,dx$

$= \int_1^{4/3} 3\,dx + \int_{4/3}^{5/3} 4\,dx + \int_{5/3}^{2} 5\,dx = 4$

2. (c): Put $e^u - 1 = t^2 \Rightarrow e^u du = 2t\,dt$

$\therefore \frac{\pi}{6} = \int_1^{\sqrt{e^x-1}} \frac{2tdt}{t(t^2+1)} = 2\left[\tan^{-1} t\right]_1^{\sqrt{e^x-1}}$

$= 2\tan^{-1}\sqrt{e^x-1} - \frac{\pi}{2}$

$\Rightarrow \tan^{-1}\sqrt{e^x-1} = \frac{\pi}{3} \Rightarrow \sqrt{e^x-1} = \tan\frac{\pi}{3} = \sqrt{3}$

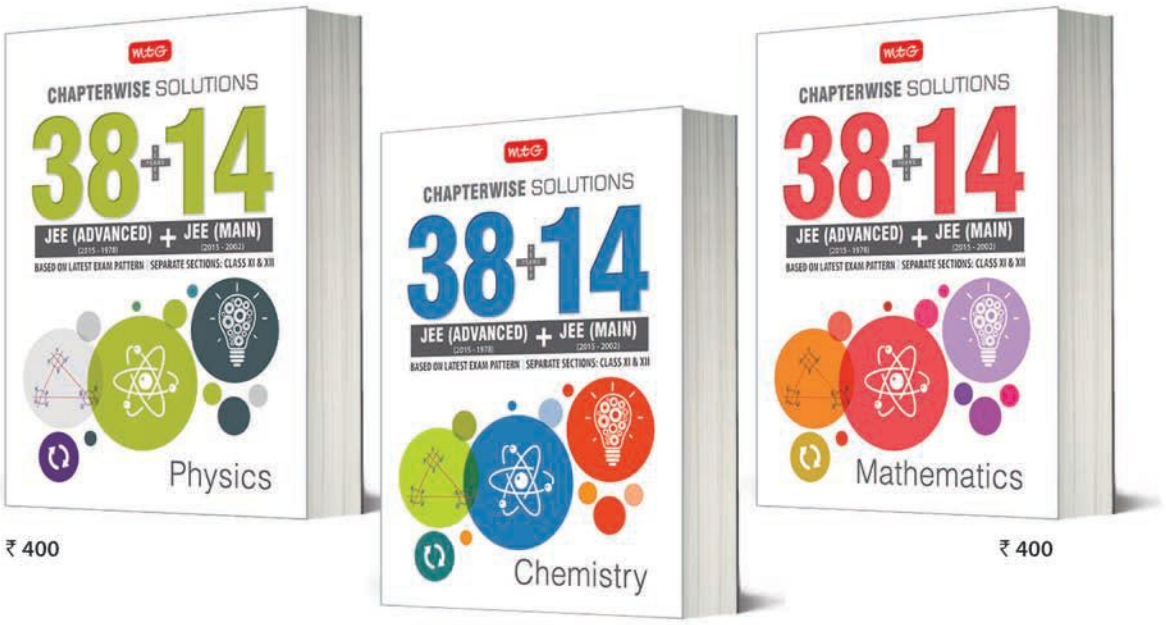
$\Rightarrow e^x = 4 \ \therefore x = \ln 4$

3. (a): Here $f'(1) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}, f'(2) = \tan\frac{\pi}{3} = \sqrt{3},$

$f'(3) = \tan\frac{\pi}{4} = 1$. Now, $\int_2^3 f'(x)\cdot f''(x)dx = \left[\frac{1}{2}\{f'(x)\}^2\right]_2^3$

By : **Prof. Shyam Bhushan,** Director, Narayana IIT Academy, Jamshedpur. Mob. : 8

Mad about rehearsing?



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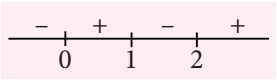
$= \frac{1}{2}\left[\{f'(3)\}^2 - \{f'(2)\}^2\right]$

and $\int_1^3 f''(x)dx = [f'(x)]_1^3 = f'(3) - f'(1)$

$\therefore$ Value $= \frac{1}{2}\left[1^2 - (\sqrt{3})^2\right] + 1 - \frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

4. **(c) :** $f'(x) = x(x^2 - 3x + 2) = x(x - 1)(x - 2)$

The sign scheme for $f'(x)$ is as below



$\therefore$ $f'(x) \le 0$ in $1 \le x \le 2$ and $f'(x) \ge 0$ in $2 \le x \le 3$. $\therefore$ $f(x)$ is decreasing in [1, 2] and increasing in [2, 3].

$\therefore$ min $f(x) = f(2) = \int_1^2 x(x^2 - 3x + 2)\,dx$

$= \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2 = -\frac{1}{4}$

max. $f(x)$ = the greatest among $\{f(1), f(3)\}$.

$f(1) = \int_1^1 x(x^2 - 3x + 2)dx = 0,\ f(3) = \int_1^3 x(x^2 - 3x + 2)dx = 2$

$\therefore$ max $f(x) = 2$, so the range $= \left[-\frac{1}{4}, 2\right]$

5. **(a) :** $\int_1^2 \left(a^2 + (4 - 4a)x + 4x^3\right)dx$

$= a^2[x]_1^2 + (2 - 2a)[x^2]_1^2 + [x^4]_1^2 = a^2 + (2 - 2a)(3) + 15$

Given, $a^2 - 6a + 21 \le 12$

$\Rightarrow$ $a^2 - 6a + 9 \le 0 \Rightarrow (a - 3)^2 \le 0$

$\Rightarrow$ $(a - 3)^2 = 0 \Rightarrow a = 3$

6. Now $\int (1 + x)\sin x\,dx = \sin x - (1 + x)\cos x + C$

$\therefore$ $I = \int \frac{(1 + x)\sin x}{[\sin x - (1 + x)\cos x]^2 - 1}dx$

$= \frac{1}{2}\ln\left|\frac{\sin x - (1 + x)\cos x - 1}{\sin x - (1 + x)\cos x + 1}\right| + C$

7. Let $I_n = \int_0^{\pi} \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin\frac{x}{2}}dx$

Then $I_{n+1} - I_n = \int_0^{\pi} \frac{\sin\left(n + \frac{3}{2}\right)x - \sin\left(n + \frac{1}{2}\right)x}{\sin\frac{x}{2}}dx$

$= 2\int_0^{\pi} \cos(n + 1)x\,dx = \frac{2}{n + 1}.[\sin(n + 1)x]_0^{\pi} = 0$

Hence $I_0 = I_1 = I_2 = I_3 = \ldots\ldots$

Thus $I_n = I_0 = \int_0^{\pi} dx = \pi$

8. The family of circles is given by

$x^2 + y^2 - ay = 0,\ a \in R$ ...(1)

Differentiating w.r.t. x, we have

$2x + 2y\frac{dy}{dx} - a\frac{dy}{dx} = 0 \Rightarrow a = 2\left(x\frac{dx}{dy} + y\right)$

Putting this value in (1), we have

$x^2 + y^2 - 2\left(x\frac{dx}{dy} + y\right)y = 0 \Rightarrow x^2 - y^2 - 2xy\frac{dx}{dy} = 0$

For the orthogonal trajectories, replace $\left(\frac{dy}{dx}\right)$ by $\left(-\frac{dx}{dy}\right)$,

to get $x^2 - y^2 + 2xy\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

This is a homogeneous differential equation in x and y.

Putting $y = vx$, we have

$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x \cdot vx} = \frac{v^2 - 1}{2v}$

$\Rightarrow$ $x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v} = -\frac{1 + v^2}{2v} \Rightarrow \frac{dx}{x} = -\frac{2v}{1 + v^2}dv$

$\Rightarrow$ $\log|x| + \log|1 + v^2| = \text{constant}$

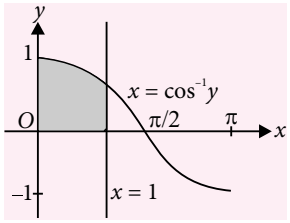
$\Rightarrow$ $|x|\,(1 + y2/x2) = \text{constant} = b$ (say)

Which is required orthogonal trajectories.

9. $x = \cos^{-1}y$

$\Rightarrow y = \cos x,\ x \in [0, \pi]$

The required area (shaded portion) is shown in the adjacent figure.



Required area $= \int_0^1 \cos x\,dx$

$= |\sin x|_0^1 = \sin 1$ sq. units.

10. The given differential equation can be re-written as

$\frac{dy}{dx}\left(y\frac{dy}{dx} - 1\right) = \left(\frac{dy}{dx} + 2\right)\left(y\frac{dy}{dx} - 1\right)$

$\Rightarrow$ $\frac{dy}{dx} = \frac{dy}{dx} + 2$ or $y\frac{dy}{dx} - 1 = 0 \Rightarrow \frac{dx}{dy} = 0$ or $y\frac{dy}{dx} = 1$

Now $\frac{dx}{dy} = 0 \Rightarrow x = c$, a family of parallel lines (lines parallel to y-axis).

$y\frac{dy}{dx} = 1 \Rightarrow$ the solution curve has subnormal unity.

The curves are $y^2 = 2x + k$, k is a constant. ■■

Engineering a new class

Despite the many engineering colleges that India boasts of, talented engineers who can really build things are a scarce commodity. But a five-year dual-degree course introduced by IIT-Madras is slowly changing and challenging the status quo. Called Engineering Design, the course aims to ingrain basic engineering concepts much beyond the duration of the course. As a result of this decade-long experiment, the graduates are making successful forays into difficult areas like biomedicine, automotives and medical technology.

"We wanted to bring engineering close to practice. Traditionally , there is a lot more theory than domain knowledge, that's not always good," said R Krishna Kumar, professor, department of engineering design at IIT-Madras.

Kumar said over the past few years, he has seen several graduates in a batch opting to set up their own ventures. For instance, there is Tarun Mehta and Swapnil Jain of Ather Energy, who took a leap to build India's first smart electric scooter. Alumni Chinmay Deodhar is building minimally invasive surgical tools, while Kartik Mehta is building the country's first sanitary napkin making machine. For a country that has seen a large number of startups in the software space, IIT-Madras is filling a crucial gap by training students with skill sets required to build hardware products. But more importantly, the course is allowing engineers to think beyond the obvious, potentially serving as a blueprint for other colleges.

"By the time we started Ather, we just knew where to look for what. We had built vehicles several times over from scratch," said Tarun Mehta, CEO of Ather, whose firm raised ₹ 75 crore from Tiger Global recently. When the department opened shop for applications in 2005, Chinmay Deodhar's well wishers advised him against opting for it.

"But the curriculum design was too good to refuse," said the 28-year old Deodhar. And for a valid reason: besides concepts of mechanical and electronics engineering, the course included a smattering of a foreign language and legal know-how. More importantly, they were also taught to appreciate various forms of art: from clay modeling, to studying prehistoric, Egyptian and Islamic art, among others. The decision probably served Deodhar well, as he now heads Croleon Innovation Labs, which is building tools to perform minimally invasive surgeries. The three-year-old firm's latest product is expected to reduce the incision size in surgeries to half a millimeter from 5 mm. While several IITs in India have a design course in their curriculum, few colleges have set up a separate department, IIT-Guwahati being one of them.

Since students are required to come up with a project every year for five years, there is a very large component of hands on learning.

> "Instead of teaching them machine tools, we allow them to play and experiment with it. So, The next time they read, they will not lose interest."
> R Krishna Kumar , Professor, Dept. of Engineering Design, IIT-Madras

Balaji Teegala recalled his course on Vehicle Dynamics, which he expected would have a lot of mathematical modeling to deal within the four walls of the classroom.Instead, the professor put a car in front of them, attached with simulators, allowing them to experiment with different controls to understand what effect it had on dynamics.

Designed to be big

Companies that have come out of IIT-Madras' Engineering Design Course

- **ATHER ENERGY-** Makes smart electric bike
- **BRUN HEALTHCARE-** Makes labour monitoring tool for expecting mothers
- **EMBRYYO TECHNOLOGIES -** Makes medical devices
- **CROLEON INNOVATION LABS-** Makes minimally invasive surgical tools
- **SARAL DESIGNS-** Makes manufacturing machines for sanitary pads

"Most of the work revolved around thinking about and coming up with new innovative things. As a student, I always thought, what is it that will create a wow-factor?" said Teegala, who is developing an electronic labour monitoring tool for expecting mothers as cofounder of Brun Healthcare.

Buoyed by the success, Krishna Kumar has decided to flip around the entire model of the course: Practicals first, to spark curiosity , followed by theory. "Instead of teaching them machine tools, we allow them to play and experiment with it. So, the next time they read, they will not lose interest," said Krishna Kumar. The course has also been able to arrest the brain drain to a certain extent as well.

"The best part is, almost everyone of us have stayed back in India. Very few have gone out of the country to study or work," said Deodhar.

Courtesy : The Economic Times

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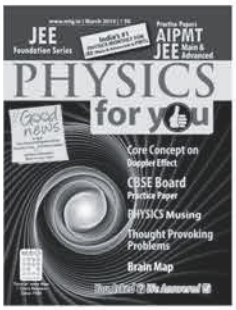

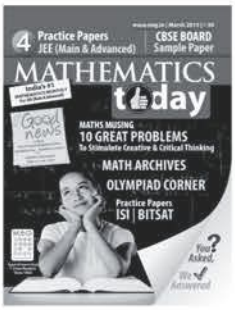

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